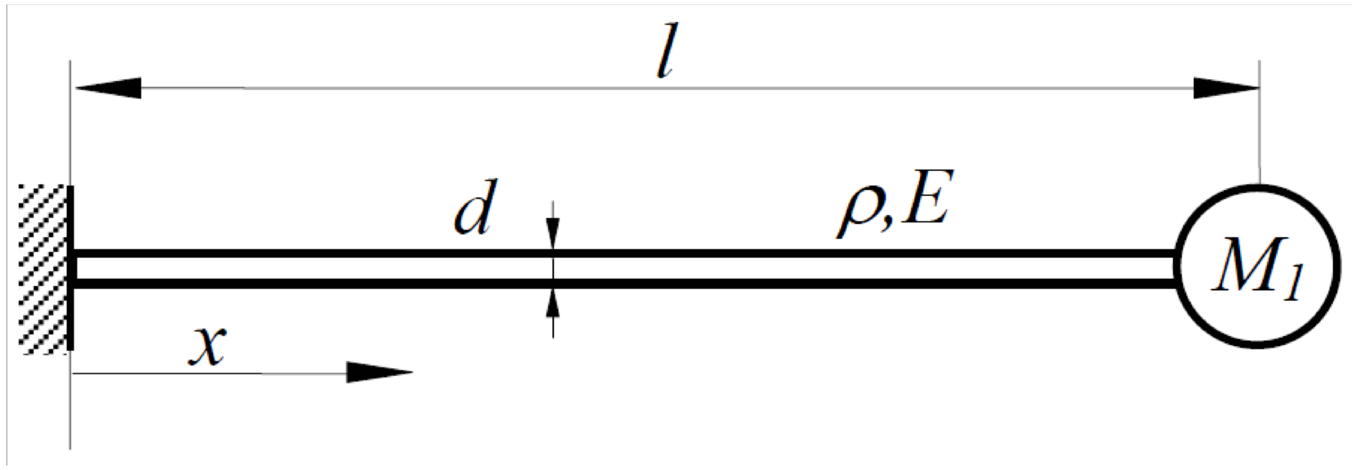
Trave incastro - massa all'estremità opposta



$Y(x) := A\cdot cos(\beta\cdot x) + B\cdot sin(\beta\cdot x) + C\cdot cosh(\beta\cdot x) + D\cdot sinh(\beta\cdot x)$

$Y'(x) := \frac{d}{dx}Y(x)$ $\qquad Y'(x)\ collect, \beta \rightarrow (B\cdot cos(\beta\cdot x) - A\cdot sin(\beta\cdot x) + C\cdot sinh(\beta\cdot x) + D\cdot cosh(\beta\cdot x))\cdot\beta$

$Y''(x) := \frac{d^2}{dx^2}Y(x)$ $\qquad Y''(x)\ collect, \beta \rightarrow (C\cdot cosh(\beta\cdot x) - B\cdot sin(\beta\cdot x) - A\cdot cos(\beta\cdot x) + D\cdot sinh(\beta\cdot x))\cdot\beta^2$

$Y'''(x) := \frac{d^3}{dx^3}Y(x)$ $\qquad Y'''(x)\ collect, \beta \rightarrow (A\cdot sin(\beta\cdot x) - B\cdot cos(\beta\cdot x) + C\cdot sinh(\beta\cdot x) + D\cdot cosh(\beta\cdot x))\cdot\beta^3$

$T(x) := E\cdot J\cdot\frac{d^3}{dx^3}Y(x)$ $\qquad$ *Azione di taglio*

$F_{inerz}(x) := -M_1\cdot\beta^4\cdot c^2\cdot Y(x)$ $\qquad$ *Forza d'inerzia*

Condizioni al contorno:

in x = 0

$Y(0) \rightarrow A + C$

$Y'(0)\ collect, \beta \rightarrow (B + D)\cdot\beta$

in x = l

$Y''(l)\ collect, \beta \rightarrow (C\cdot cosh(\beta\cdot l) - B\cdot sin(\beta\cdot l) - A\cdot cos(\beta\cdot l) + D\cdot sinh(\beta\cdot l))\cdot\beta^2$

$T(l) - F_{inerz}(l)\ collect, A, B, C, D \rightarrow \left(M_1\cdot\beta^4\cdot c^2\cdot cos(\beta\cdot l) + E\cdot J\cdot\beta^3\cdot sin(\beta\cdot l)\right)\cdot A + \left(M_1\cdot\beta^4\cdot c^2\cdot sin(\beta\cdot l) - E\cdot J\cdot\beta^3\cdot cos(\beta\cdot l)\right)\cdot B + \left(M_1\cdot\beta^4\cdot c^2\cdot cosh(\beta\cdot l) + E\cdot J\cdot\beta^3\cdot sinh(\beta\cdot l)\right)\cdot C + \left(M_1\cdot\beta^4\cdot c^2\cdot sinh(\beta\cdot l) + E\cdot J\cdot\beta^3\cdot cosh(\beta\cdot l)\right)\cdot D$

Riassunto delle condizioni al contorno (che formano un sistema lineare)

$A + C = 0$

$B + D = 0$

$C \cdot cosh(\beta \cdot l) - B \cdot sin(\beta \cdot l) - A \cdot cos(\beta \cdot l) + D \cdot sinh(\beta \cdot l) = 0$

$$\left(M_I \cdot \beta^4 \cdot c^2 \cdot cos(\beta \cdot l) + E \cdot J \cdot \beta^3 \cdot sin(\beta \cdot l)\right) \cdot A + \left(M_I \cdot \beta^4 \cdot c^2 \cdot sin(\beta \cdot l) - E \cdot J \cdot \beta^3 \cdot cos(\beta \cdot l)\right) \cdot B + \left(M_I \cdot \beta^4 \cdot c^2 \cdot cosh(\beta \cdot l) + E \cdot J \cdot \beta^3 \cdot sinh(\beta \cdot l)\right) \cdot C + \left(M_I \cdot \beta^4 \cdot c^2 \cdot sinh(\beta \cdot l) + E \cdot J \cdot \beta^3 \cdot cosh(\beta \cdot l)\right) \cdot D = 0$$

in questa equazione si può semplificare β^3

Poniamo per semplicità: $\alpha = \beta\, l$ e otteniamo:

$A + C = 0$

$B + D = 0$

$C \cdot cosh(\alpha) - B \cdot sin(\alpha) - A \cdot cos(\alpha) + D \cdot sinh(\alpha) = 0$

$$\left[M_I \cdot \left(\frac{\alpha}{l}\right) \cdot c^2 \cdot cos(\alpha) + E \cdot J \cdot sin(\alpha)\right] \cdot A + \left[M_I \cdot \left(\frac{\alpha}{l}\right) \cdot c^2 \cdot sin(\alpha) - E \cdot J \cdot cos(\alpha)\right] \cdot B + \left[M_I \cdot \left(\frac{\alpha}{l}\right) \cdot c^2 \cdot cosh(\alpha) + E \cdot J \cdot sinh(\alpha)\right] \cdot C + \left[M_I \cdot \left(\frac{\alpha}{l}\right) \cdot c^2 \cdot sinh(\alpha) + E \cdot J \cdot cosh(\alpha)\right] \cdot D = 0$$

dove $c^2 = \dfrac{E \cdot J}{\rho \cdot A_{sez}}$

Quest'ultima condizione, dopo alcune semplificazioni si può riscrivere nella forma:

$$\left[M_I \cdot \left(\frac{\alpha}{l}\right) \cdot \frac{1}{\rho \cdot A_{sez}} \cdot cos(\alpha) + sin(\alpha)\right] \cdot A + \left[M_I \cdot \left(\frac{\alpha}{l}\right) \cdot \frac{1}{\rho \cdot A_{sez}} \cdot sin(\alpha) - cos(\alpha)\right] \cdot B + \left[M_I \cdot \left(\frac{\alpha}{l}\right) \cdot \frac{1}{\rho \cdot A_{sez}} \cdot cosh(\alpha) + sinh(\alpha)\right] \cdot C + \left[M_I \cdot \left(\frac{\alpha}{l}\right) \cdot \frac{1}{\rho \cdot A_{sez}} \cdot sinh(\alpha) + cosh(\alpha)\right] \cdot D = 0$$

In tutti i termini compare il rapporto: $\lambda = \dfrac{M_I}{\rho \cdot A_{sez} \cdot l}$ *che fornisce il rapporto tra la massa concentrata e la massa totale della trave: pertanto si ha:*

$$(\lambda \cdot \alpha \cdot cos(\alpha) + sin(\alpha)) \cdot A + (\lambda \cdot \alpha \cdot sin(\alpha) - cos(\alpha)) \cdot B + (\lambda \cdot \alpha \cdot cosh(\alpha) + sinh(\alpha)) \cdot C + (\lambda \cdot \alpha \cdot sinh(\alpha) + cosh(\alpha)) \cdot D = 0$$

ovvero:

$$f_A(\alpha) \cdot A + f_B(\alpha) \cdot B + f_C(\alpha) \cdot C + f_D(\alpha) \cdot D = 0$$

dove si è posto:

$f_A(\alpha) := \lambda \cdot \alpha \cdot cos(\alpha) + sin(\alpha)$

$f_B(\alpha) := \lambda \cdot \alpha \cdot sin(\alpha) - cos(\alpha)$

$f_C(\alpha) := \lambda \cdot \alpha \cdot cosh(\alpha) + sinh(\alpha)$

$f_D(\alpha) := \lambda \cdot \alpha \cdot sinh(\alpha) + cosh(\alpha)$

Le quattro condizioni al contorno si possono scrivere in forma matriciale, come segue

$$\Delta(\alpha)\cdot\begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Delta(\alpha) := \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -cos(\alpha) & -sin(\alpha) & cosh(\alpha) & sinh(\alpha) \\ f_A(\alpha) & f_B(\alpha) & f_C(\alpha) & f_D(\alpha) \end{pmatrix}$$

Conviene procedere per via numerica, senza sviluppi simbolici...

Lunghezza

$l := 0.8$

Materiale: acciaio

$\rho := 7800$

$E := 206000\cdot 10^{6} = 2.06\times 10^{11}$

Sezione circolare di diametro d:

$d := 25\cdot 10^{-3} = 0.025$

$J := \dfrac{\pi\cdot d^4}{64} = 1.917\times 10^{-8}$

$A_{sez} := \dfrac{\pi\cdot d^2}{4} = 4.909\times 10^{-4}$

Massa concentrata

$M_1 := 15$

$m_{trave} := \rho\cdot A_{sez}\cdot l = 3.063$

$\lambda := \dfrac{M_1}{\rho\cdot A_{sez}\cdot l} = 4.897$ $\qquad$ $c := \sqrt{\dfrac{E\cdot J}{\rho\cdot A_{sez}}} = 32.119$

$f_A(\alpha) := \lambda \cdot \alpha \cdot cos(\alpha) + sin(\alpha)$

$f_B(\alpha) := \lambda \cdot \alpha \cdot sin(\alpha) - cos(\alpha)$

$f_C(\alpha) := \lambda \cdot \alpha \cdot cosh(\alpha) + sinh(\alpha)$

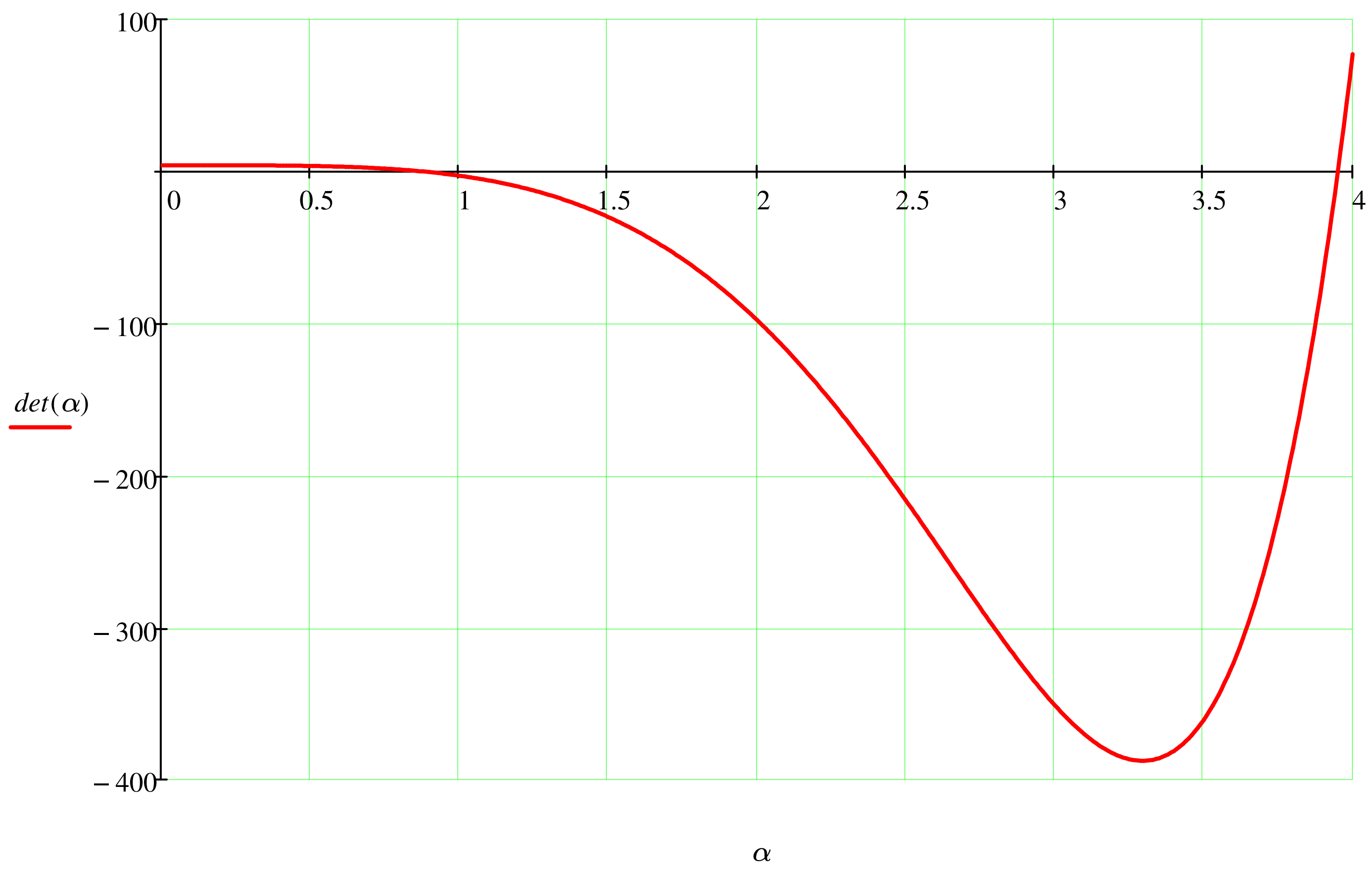
$f_D(\alpha) := \lambda \cdot \alpha \cdot sinh(\alpha) + cosh(\alpha)$

$$\Delta(\alpha) := \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -cos(\alpha) & -sin(\alpha) & cosh(\alpha) & sinh(\alpha) \\ f_A(\alpha) & f_B(\alpha) & f_C(\alpha) & f_D(\alpha) \end{pmatrix}$$

Cerchiamo le soluzioni dell'equazione caratteristica per via grafica:

$det(\alpha) := |\Delta(\alpha)|$

$\alpha := 0, 0.01 .. 4$



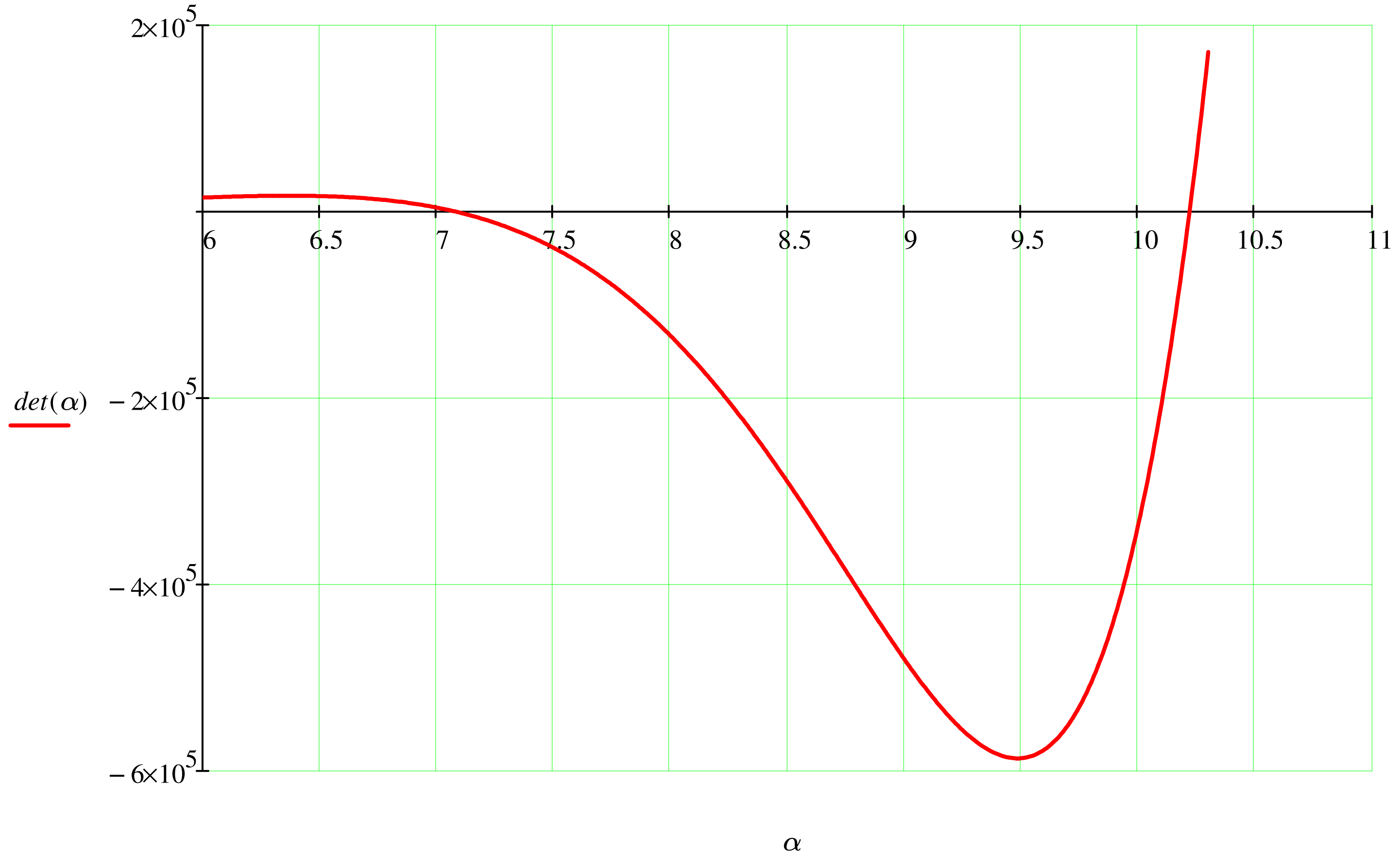
$\alpha := 2$

$\alpha_1 := root(det(\alpha), \alpha) = 0.87435$

$\alpha := 4.6$

$\alpha_2 := root(det(\alpha), \alpha) = 3.950457$

$\alpha := 6, 6.01 .. 10.3$



$\alpha := 7.85$

$\alpha_3 := root(det(\alpha), \alpha) = 7.082826$

$\alpha := 11$

$\alpha_4 := root(det(\alpha), \alpha) = 10.220066$

$\alpha = \beta \cdot l$ $\qquad \beta = \dfrac{\alpha}{l}$

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \begin{pmatrix} 0.87435 \\ 3.950457 \\ 7.082826 \\ 10.220066 \end{pmatrix}$$

$$\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix} := \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} \cdot \frac{1}{l} = \begin{pmatrix} 1.092938 \\ 4.938071 \\ 8.853533 \\ 12.775083 \end{pmatrix}$$

$\omega = \beta^2 \cdot c$

$\omega_1 := (\beta_1)^2 \cdot c = 38.367$ $\qquad f_1 := \dfrac{\omega_1}{2 \cdot \pi} = 6.106$

$\omega_2 := (\beta_2)^2 \cdot c = 783.215$ $\qquad f_2 := \dfrac{\omega_2}{2 \cdot \pi} = 124.653$

$\omega_3 := (\beta_3)^2 \cdot c = 2517.674$ $\qquad f_3 := \dfrac{\omega_3}{2 \cdot \pi} = 400.7$

$\omega_4 := (\beta_4)^2 \cdot c = 5241.961$ $\qquad f_4 := \dfrac{\omega_4}{2 \cdot \pi} = 834.284$

$l := l$

Given

$A + C = 0$

$B + D = 0$

$C \cdot cosh(\beta \cdot l) - B \cdot sin(\beta \cdot l) - A \cdot cos(\beta \cdot l) + D \cdot sinh(\beta \cdot l) = 0$

$$Find(B, C, D) \rightarrow \begin{pmatrix} -\dfrac{A \cdot cos(\beta \cdot l) + A \cdot cosh(\beta \cdot l)}{sinh(\beta \cdot l) + sin(\beta \cdot l)} \\ -A \\ \dfrac{A \cdot cos(\beta \cdot l) + A \cdot cosh(\beta \cdot l)}{sinh(\beta \cdot l) + sin(\beta \cdot l)} \end{pmatrix}$$

$A := 1$

$$B(\beta) := -\frac{A \cdot cos(\beta \cdot l) + A \cdot cosh(\beta \cdot l)}{sinh(\beta \cdot l) + sin(\beta \cdot l)}$$

$C := -A$

$$D(\beta) := \frac{A \cdot cos(\beta \cdot l) + A \cdot cosh(\beta \cdot l)}{sinh(\beta \cdot l) + sin(\beta \cdot l)}$$

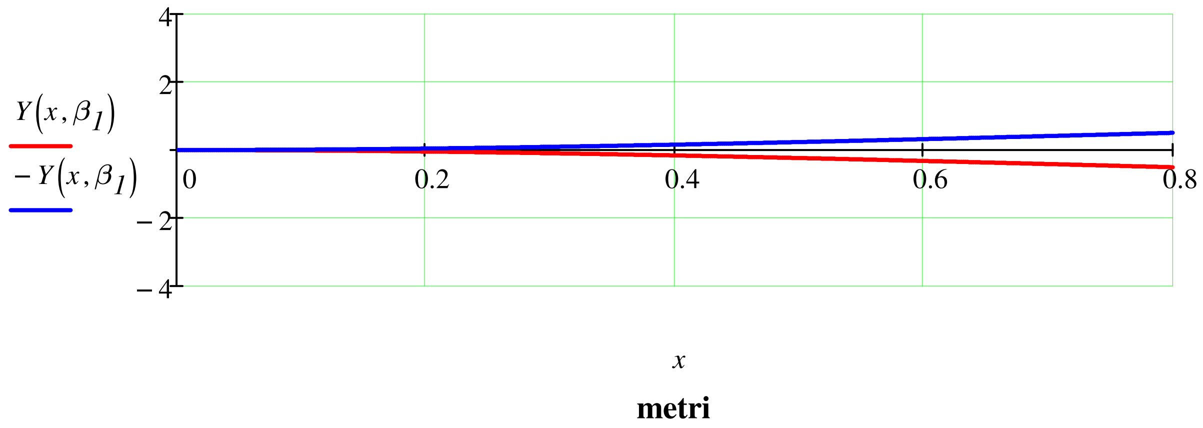
$Y(x, \beta) := A \cdot cos(\beta \cdot x) + B(\beta) \cdot sin(\beta \cdot x) + C \cdot cosh(\beta \cdot x) + D(\beta) \cdot sinh(\beta \cdot x)$

$$x := 0, \frac{l}{500} .. l$$

$FS := 4$ *Fondo scala per i grafici*

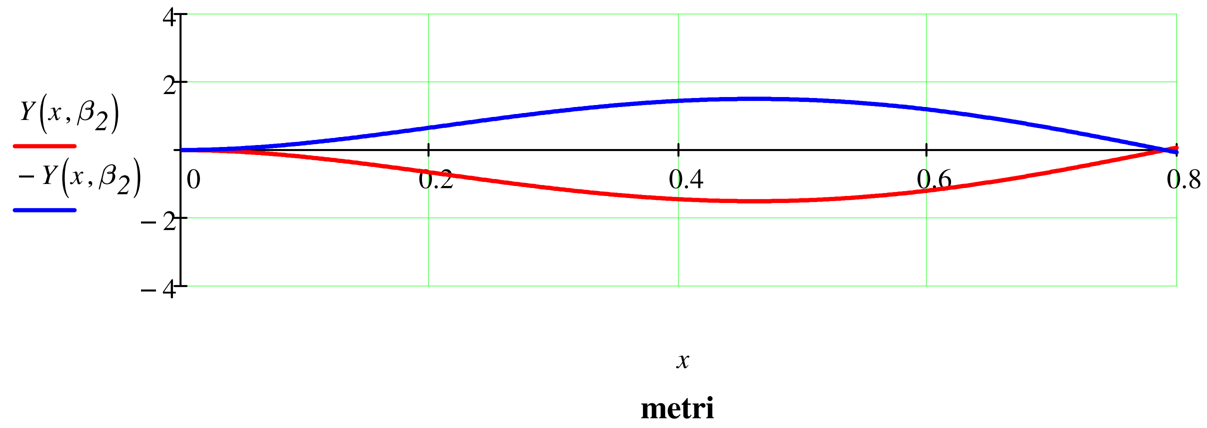
$f_1 = 6.106\, s \cdot Hz$

Mode shape (1)



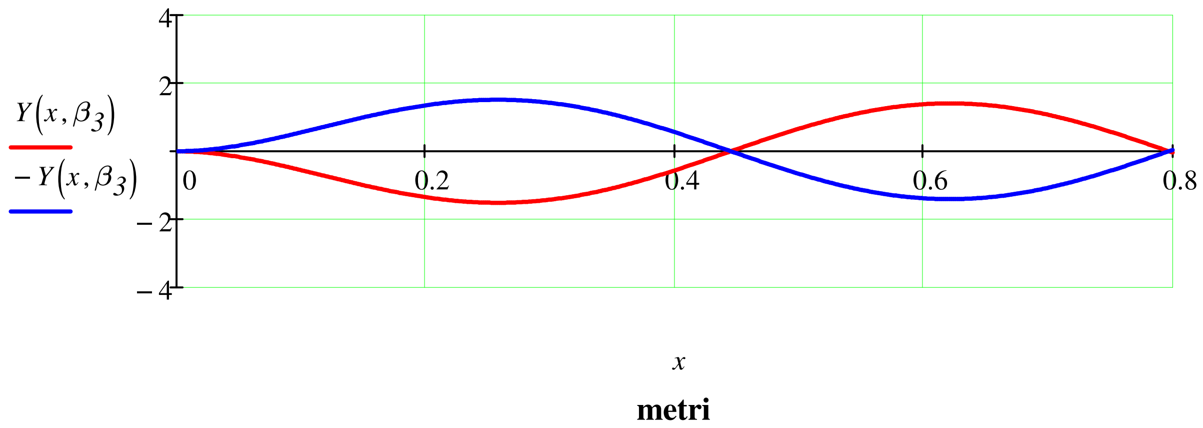
$f_2 = 124.653\, s \cdot Hz$

Mode shape (2)



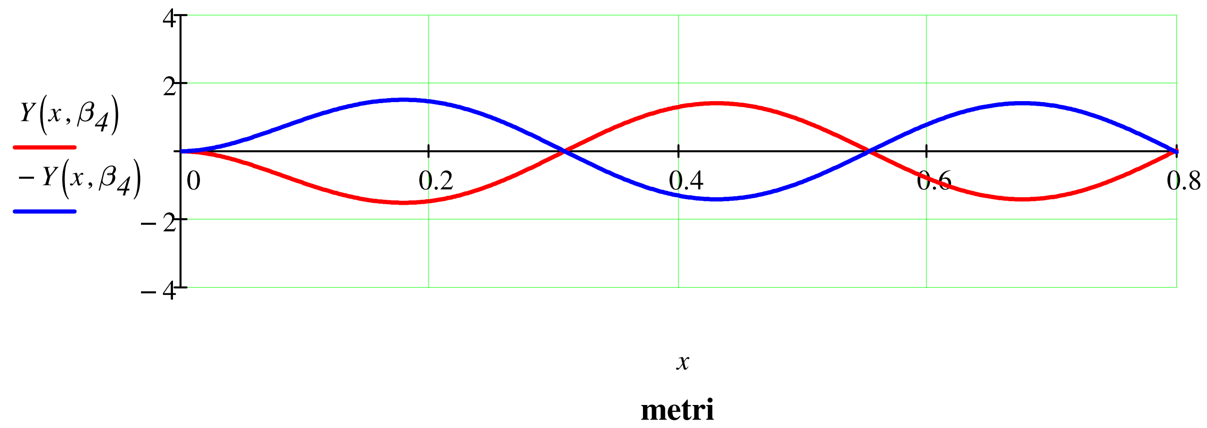
$f_3 = 400.7\, s \cdot Hz$

Mode shape (3)



$f_4 = 834.284\, s \cdot Hz$

Mode shape (4)



Animazioni delle deformate (Usare il menù: Strumenti - Aminazione - Registra)