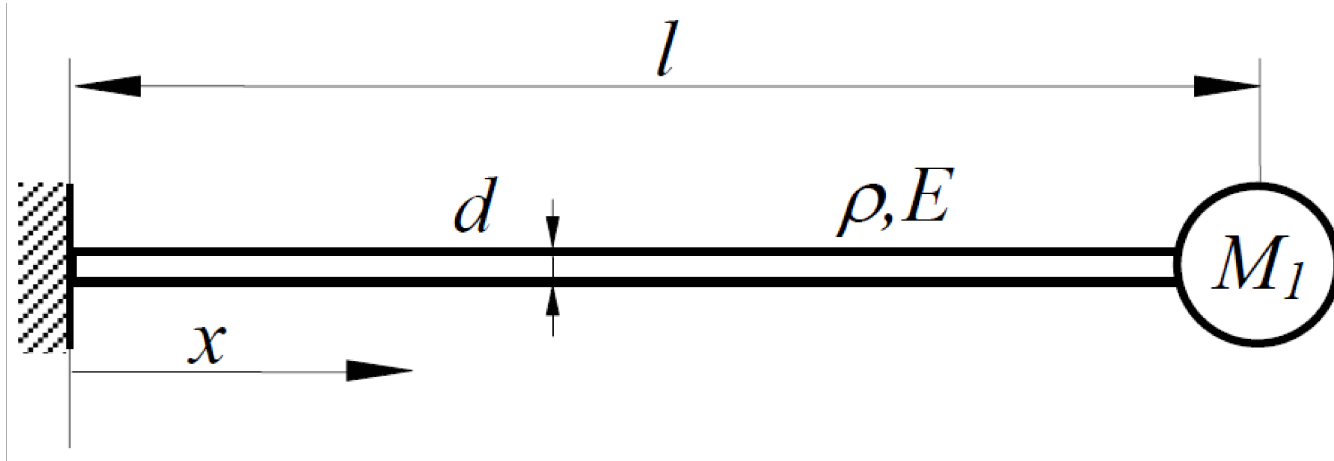


Trave incastro - massa all'estremità opposta



$$Y(x) := A \cdot \cos(\beta \cdot x) + B \cdot \sin(\beta \cdot x) + C \cdot \cosh(\beta \cdot x) + D \cdot \sinh(\beta \cdot x)$$

$$Y'(x) := \frac{d}{dx} Y(x) \quad Y'(x) \text{ collect } \beta \rightarrow (B \cdot \cos(\beta \cdot x) - A \cdot \sin(\beta \cdot x) + C \cdot \sinh(\beta \cdot x) + D \cdot \cosh(\beta \cdot x)) \cdot \beta$$

$$Y''(x) := \frac{d^2}{dx^2} Y(x) \quad Y''(x) \text{ collect } \beta \rightarrow (C \cdot \cosh(\beta \cdot x) - B \cdot \sin(\beta \cdot x) - A \cdot \cos(\beta \cdot x) + D \cdot \sinh(\beta \cdot x)) \cdot \beta^2$$

$$Y'''(x) := \frac{d^3}{dx^3} Y(x) \quad Y'''(x) \text{ collect } \beta \rightarrow (A \cdot \sin(\beta \cdot x) - B \cdot \cos(\beta \cdot x) + C \cdot \sinh(\beta \cdot x) + D \cdot \cosh(\beta \cdot x)) \cdot \beta^3$$

$$T(x) := E \cdot J \cdot \frac{d^3}{dx^3} Y(x) \quad \text{Azione di taglio}$$

$$F_{inerz}(x) := -M_I \cdot \beta^4 \cdot c^2 \cdot Y(x) \quad \text{Forza d'inerzia}$$

Condizioni al contorno:

in $x = 0$

$$Y(0) \rightarrow A + C$$

$$Y'(0) \text{ collect } \beta \rightarrow (B + D) \cdot \beta$$

in $x = l$

$$Y''(l) \text{ collect } \beta \rightarrow (C \cdot \cosh(\beta \cdot l) - B \cdot \sin(\beta \cdot l) - A \cdot \cos(\beta \cdot l) + D \cdot \sinh(\beta \cdot l)) \cdot \beta^2$$

$$T(l) - F_{inerz}(l) \text{ collect } A, B, C, D \rightarrow \left(M_I \cdot \beta^4 \cdot c^2 \cdot \cos(\beta \cdot l) + E \cdot J \cdot \beta^3 \cdot \sin(\beta \cdot l) \right) \cdot A + \left(M_I \cdot \beta^4 \cdot c^2 \cdot \sin(\beta \cdot l) - E \cdot J \cdot \beta^3 \cdot \cos(\beta \cdot l) \right) \cdot B + \left(M_I \cdot \beta^4 \cdot c^2 \cdot \cosh(\beta \cdot l) + E \cdot J \cdot \beta^3 \cdot \sinh(\beta \cdot l) \right) \cdot C + \left(M_I \cdot \beta^4 \cdot c^2 \cdot \sinh(\beta \cdot l) + E \cdot J \cdot \beta^3 \cdot \cosh(\beta \cdot l) \right) \cdot D$$

Riassunto delle condizioni al contorno (che formano un sistema lineare)

$$A + C = 0$$

$$B + D = 0$$

$$C \cdot \cosh(\beta \cdot l) - B \cdot \sin(\beta \cdot l) - A \cdot \cos(\beta \cdot l) + D \cdot \sinh(\beta \cdot l) = 0$$

$$\left(M_I \cdot \beta^4 \cdot c^2 \cdot \cos(\beta \cdot l) + E \cdot J \cdot \beta^3 \cdot \sin(\beta \cdot l) \right) \cdot A + \left(M_I \cdot \beta^4 \cdot c^2 \cdot \sin(\beta \cdot l) - E \cdot J \cdot \beta^3 \cdot \cos(\beta \cdot l) \right) \cdot B + \left(M_I \cdot \beta^4 \cdot c^2 \cdot \cosh(\beta \cdot l) + E \cdot J \cdot \beta^3 \cdot \sinh(\beta \cdot l) \right) \cdot C + \left(M_I \cdot \beta^4 \cdot c^2 \cdot \sinh(\beta \cdot l) + E \cdot J \cdot \beta^3 \cdot \cosh(\beta \cdot l) \right) \cdot D = 0$$

in questa equazione si può semplificare β^3

Poniamo per semplicità: $\alpha = \beta l$ e otteniamo:

$$A + C = 0$$

$$B + D = 0$$

$$C \cdot \cosh(\alpha) - B \cdot \sin(\alpha) - A \cdot \cos(\alpha) + D \cdot \sinh(\alpha) = 0$$

$$\left[M_I \cdot \left(\frac{\alpha}{l} \right) \cdot c^2 \cdot \cos(\alpha) + E \cdot J \cdot \sin(\alpha) \right] \cdot A + \left[M_I \cdot \left(\frac{\alpha}{l} \right) \cdot c^2 \cdot \sin(\alpha) - E \cdot J \cdot \cos(\alpha) \right] \cdot B + \left[M_I \cdot \left(\frac{\alpha}{l} \right) \cdot c^2 \cdot \cosh(\alpha) + E \cdot J \cdot \sinh(\alpha) \right] \cdot C + \left[M_I \cdot \left(\frac{\alpha}{l} \right) \cdot c^2 \cdot \sinh(\alpha) + E \cdot J \cdot \cosh(\alpha) \right] \cdot D = 0$$

dove $c^2 = \frac{E \cdot J}{\rho \cdot A_{sez}}$

Quest'ultima condizione, dopo alcune semplificazioni si può riscrivere nella forma:

$$\left[M_I \cdot \left(\frac{\alpha}{l} \right) \cdot \frac{1}{\rho \cdot A_{sez}} \cdot \cos(\alpha) + \sin(\alpha) \right] \cdot A + \left[M_I \cdot \left(\frac{\alpha}{l} \right) \cdot \frac{1}{\rho \cdot A_{sez}} \cdot \sin(\alpha) - \cos(\alpha) \right] \cdot B + \left[M_I \cdot \left(\frac{\alpha}{l} \right) \cdot \frac{1}{\rho \cdot A_{sez}} \cdot \cosh(\alpha) + \sinh(\alpha) \right] \cdot C + \left[M_I \cdot \left(\frac{\alpha}{l} \right) \cdot \frac{1}{\rho \cdot A_{sez}} \cdot \sinh(\alpha) + \cosh(\alpha) \right] \cdot D = 0$$

In tutti i termini compare il rapporto: $\lambda = \frac{M_I}{\rho \cdot A_{sez} \cdot l}$ che fornisce il rapporto tra la massa concentrata e la massa totale della trave: pertanto si ha:

$$(\lambda \cdot \alpha \cdot \cos(\alpha) + \sin(\alpha)) \cdot A + (\lambda \cdot \alpha \cdot \sin(\alpha) - \cos(\alpha)) \cdot B + (\lambda \cdot \alpha \cdot \cosh(\alpha) + \sinh(\alpha)) \cdot C + (\lambda \cdot \alpha \cdot \sinh(\alpha) + \cosh(\alpha)) \cdot D = 0$$

ovvero:

$$f_A(\alpha) \cdot A + f_B(\alpha) \cdot B + f_C(\alpha) \cdot C + f_D(\alpha) \cdot D = 0$$

dove si è posto:

$$f_A(\alpha) := \lambda \cdot \alpha \cdot \cos(\alpha) + \sin(\alpha)$$

$$f_B(\alpha) := \lambda \cdot \alpha \cdot \sin(\alpha) - \cos(\alpha)$$

$$f_C(\alpha) := \lambda \cdot \alpha \cdot \cosh(\alpha) + \sinh(\alpha)$$

$$f_D(\alpha) := \lambda \cdot \alpha \cdot \sinh(\alpha) + \cosh(\alpha)$$

Le quattro condizioni al contorno si possono scrivere in forma matriciale, come segue

$$\Delta(\alpha) \cdot \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Delta(\alpha) := \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -\cos(\alpha) & -\sin(\alpha) & \cosh(\alpha) & \sinh(\alpha) \\ f_A(\alpha) & f_B(\alpha) & f_C(\alpha) & f_D(\alpha) \end{pmatrix}$$

Conviene procedere per via numerica, senza sviluppi simbolici...

Lunghezza

$$l_w := 0.8$$

Materiale: acciaio

$$\rho := 7800$$

$$E := 206000 \cdot 10^6 = 2.06 \times 10^{11}$$

Sezione circolare di diametro d:

$$d := 25 \cdot 10^{-3} = 0.025$$

$$J_w := \frac{\pi \cdot d^4}{64} = 1.917 \times 10^{-8}$$

$$A_{sez} := \frac{\pi \cdot d^2}{4} = 4.909 \times 10^{-4}$$

Massa concentrata

$$M_I := 15$$

$$m_{trave} := \rho \cdot A_{sez} \cdot l = 3.063$$

$$\lambda := \frac{M_I}{\rho \cdot A_{sez} \cdot l} = 4.897$$



$$\underline{\alpha} := \sqrt{\frac{E \cdot J}{\rho \cdot A_{sez}}} = 32.119$$

$$f_A(\alpha) := \lambda \cdot \alpha \cdot \cos(\alpha) + \sin(\alpha)$$

$$f_B(\alpha) := \lambda \cdot \alpha \cdot \sin(\alpha) - \cos(\alpha)$$

$$f_C(\alpha) := \lambda \cdot \alpha \cdot \cosh(\alpha) + \sinh(\alpha)$$

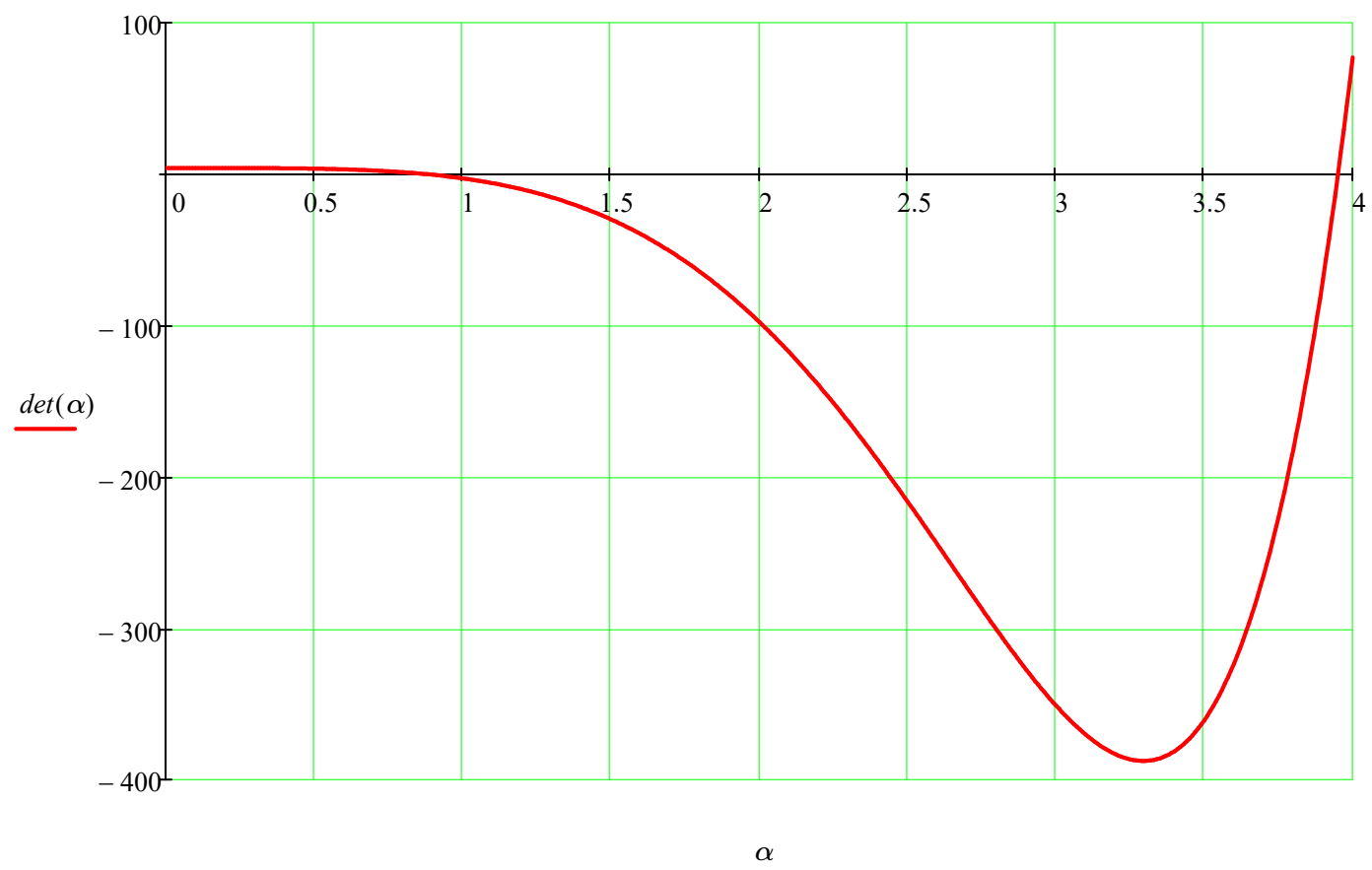
$$f_D(\alpha) := \lambda \cdot \alpha \cdot \sinh(\alpha) + \cosh(\alpha)$$

$$\Delta(\alpha) := \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -\cos(\alpha) & -\sin(\alpha) & \cosh(\alpha) & \sinh(\alpha) \\ f_A(\alpha) & f_B(\alpha) & f_C(\alpha) & f_D(\alpha) \end{pmatrix}$$

Cerchiamo le soluzioni dell'equazione caratteristica per via grafica:

$$\det(\alpha) := |\Delta(\alpha)|$$

$$\alpha := 0, 0.01 .. 4$$



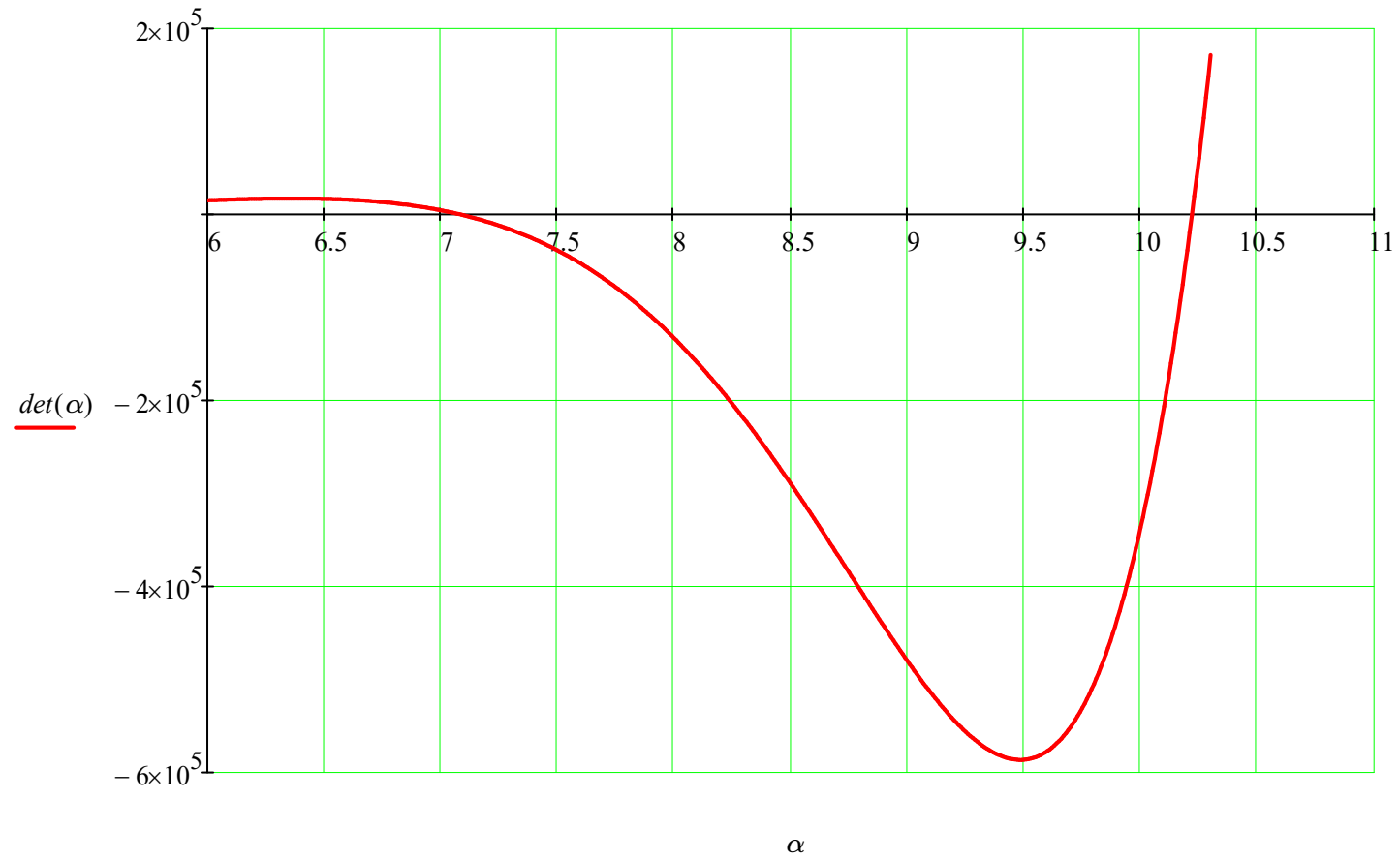
$$\alpha := 2$$

$$\alpha_1 := root(det(\alpha), \alpha) = 0.87435$$

$$\alpha := 4.6$$

$$\alpha_2 := root(det(\alpha), \alpha) = 3.950457$$

$$\alpha := 6, 6.01 .. 10.3$$



$$\alpha := 7.85$$

$$\alpha_3 := root(det(\alpha), \alpha) = 7.082826$$

$$\alpha := 11$$

$$\alpha_4 := root(det(\alpha), \alpha) = 10.220066$$

$$\alpha = \beta \cdot l$$

$$\beta = \frac{\alpha}{l}$$

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \begin{pmatrix} 0.87435 \\ 3.950457 \\ 7.082826 \\ 10.220066 \end{pmatrix}$$

$$\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix} := \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} \cdot \frac{1}{l} = \begin{pmatrix} 1.092938 \\ 4.938071 \\ 8.853533 \\ 12.775083 \end{pmatrix}$$

$$\omega = \beta^2 \cdot c$$

$$\omega_1 := (\beta_1)^2 \cdot c = 38.367$$

$$f_1 := \frac{\omega_1}{2 \cdot \pi} = 6.106$$

$$\omega_2 := (\beta_2)^2 \cdot c = 783.215$$

$$f_2 := \frac{\omega_2}{2 \cdot \pi} = 124.653$$

$$\omega_3 := (\beta_3)^2 \cdot c = 2517.674$$

$$f_3 := \frac{\omega_3}{2 \cdot \pi} = 400.7$$

$$\omega_4 := (\beta_4)^2 \cdot c = 5241.961$$

$$f_4 := \frac{\omega_4}{2 \cdot \pi} = 834.284$$

$$l := l$$

Given

$$A + C = 0$$

$$B + D = 0$$

$$C \cdot \cosh(\beta \cdot l) - B \cdot \sin(\beta \cdot l) - A \cdot \cos(\beta \cdot l) + D \cdot \sinh(\beta \cdot l) = 0$$

$$\text{Find}(B, C, D) \rightarrow \begin{pmatrix} -\frac{A \cdot \cos(\beta \cdot l) + A \cdot \cosh(\beta \cdot l)}{\sinh(\beta \cdot l) + \sin(\beta \cdot l)} \\ -A \\ \frac{A \cdot \cos(\beta \cdot l) + A \cdot \cosh(\beta \cdot l)}{\sinh(\beta \cdot l) + \sin(\beta \cdot l)} \end{pmatrix}$$

$$A := 1$$

$$B(\beta) := -\frac{A \cdot \cos(\beta \cdot l) + A \cdot \cosh(\beta \cdot l)}{\sinh(\beta \cdot l) + \sin(\beta \cdot l)}$$

$$C := -A$$

$$D(\beta) := \frac{A \cdot \cos(\beta \cdot l) + A \cdot \cosh(\beta \cdot l)}{\sinh(\beta \cdot l) + \sin(\beta \cdot l)}$$

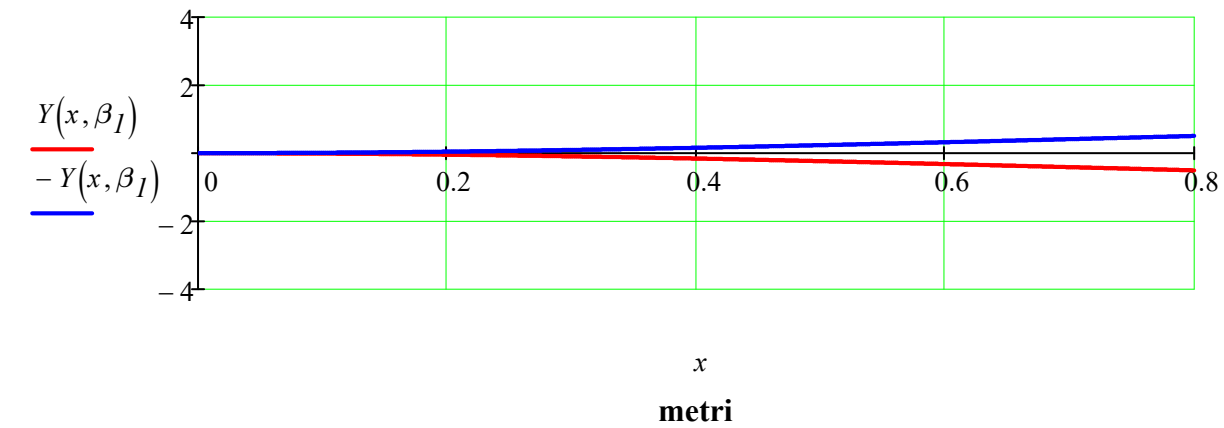
$$Y(x, \beta) := A \cdot \cos(\beta \cdot x) + B(\beta) \cdot \sin(\beta \cdot x) + C \cdot \cosh(\beta \cdot x) + D(\beta) \cdot \sinh(\beta \cdot x)$$

$$x := 0, \frac{l}{500} .. l$$

FS := 4 Fondo scala per i grafici

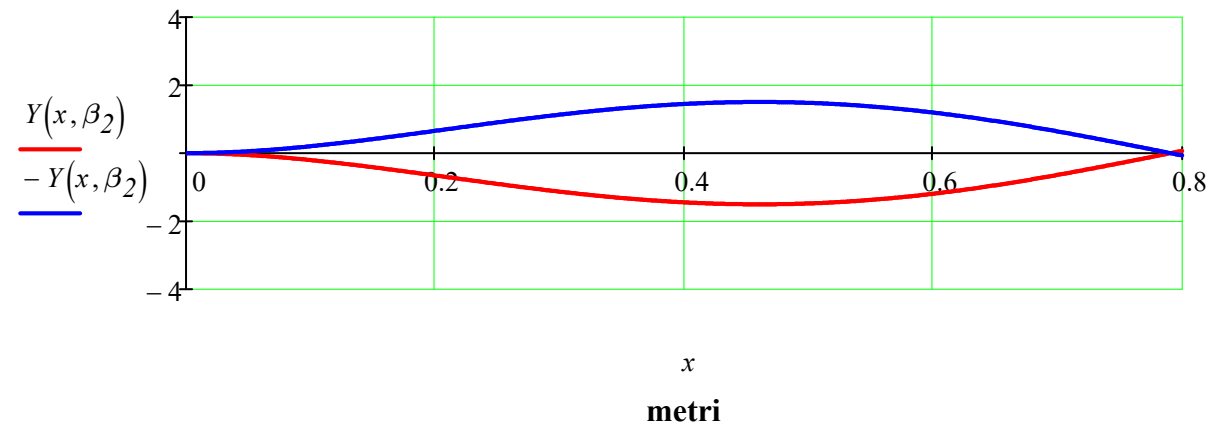
$$f_1 = 6.106 \text{ s}\cdot\text{Hz}$$

Mode shape (1)



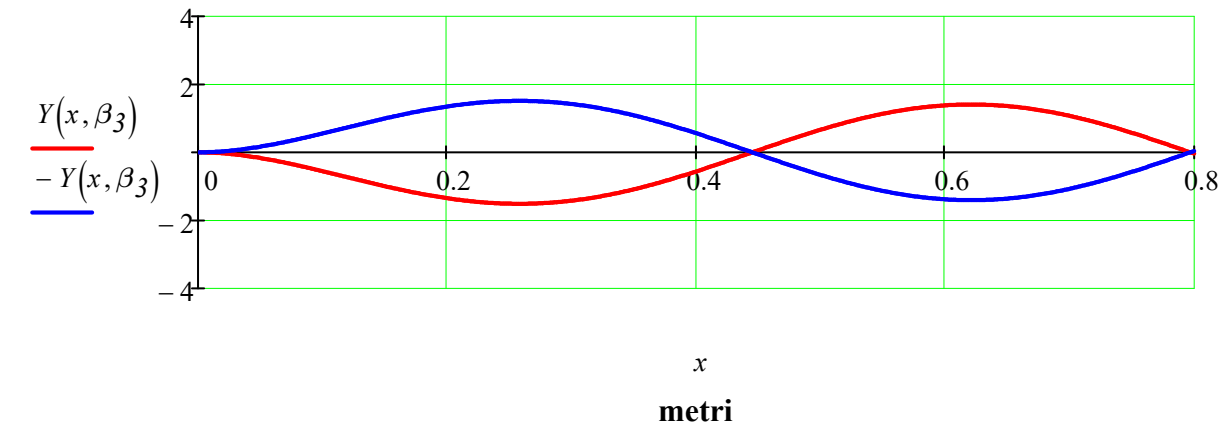
$$f_2 = 124.653 \text{ s}\cdot\text{Hz}$$

Mode shape (2)



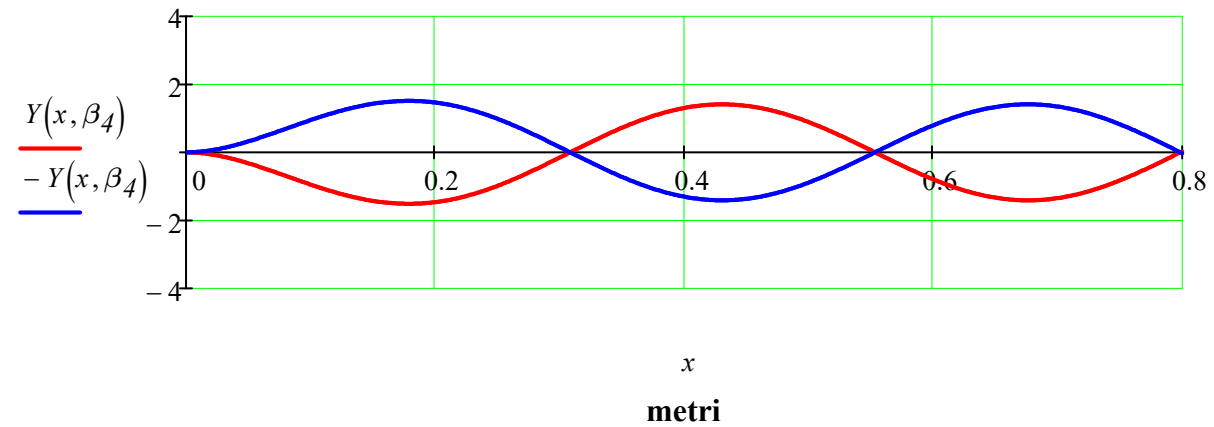
$$f_3 = 400.7 \text{ s}\cdot\text{Hz}$$

Mode shape (3)



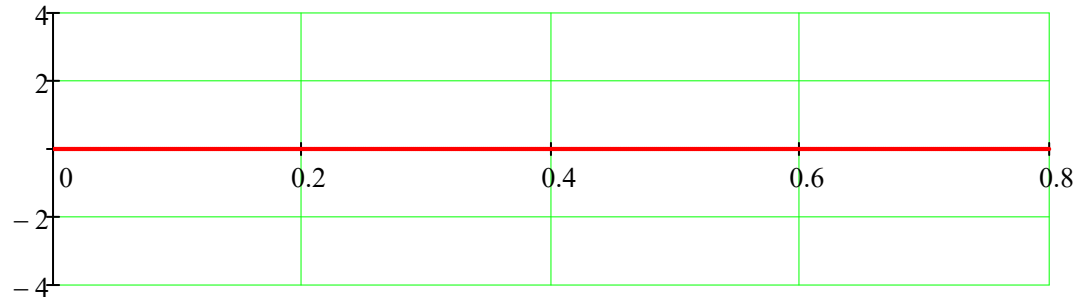
$$f_4 = 834.284 \text{ s}\cdot\text{Hz}$$

Mode shape (4)



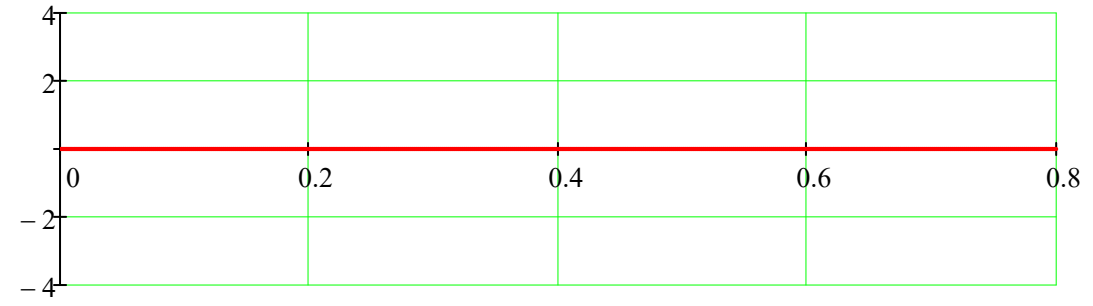
Animazioni delle deformate (Usare il menù: Strumenti - Animazione - Registra)

Mode shape (1)



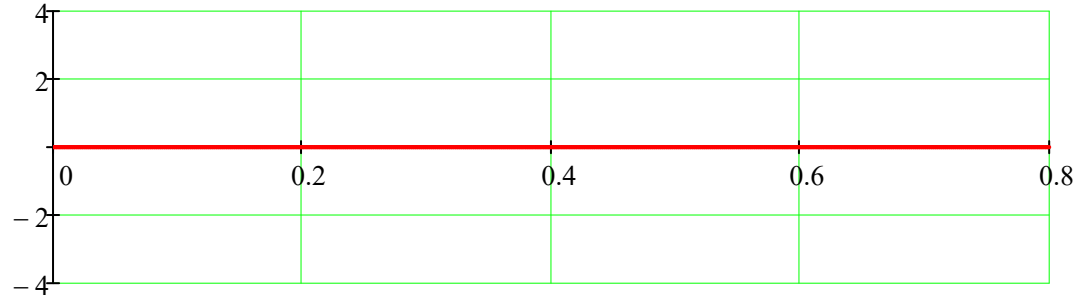
metri

Mode shape (2)



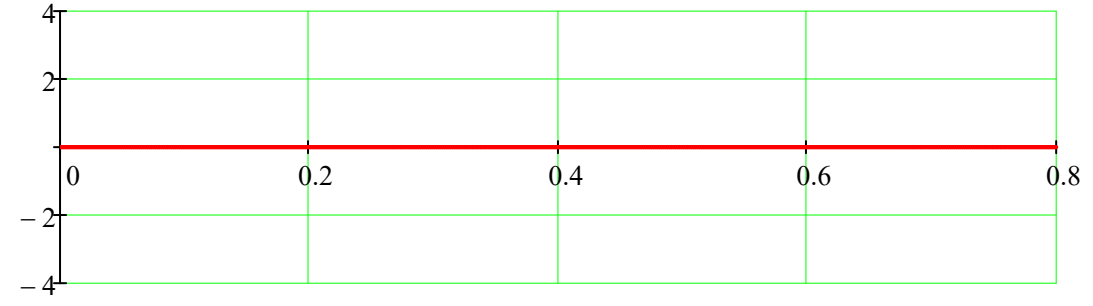
metri

Mode shape (3)



metri

Mode shape (4)



metri