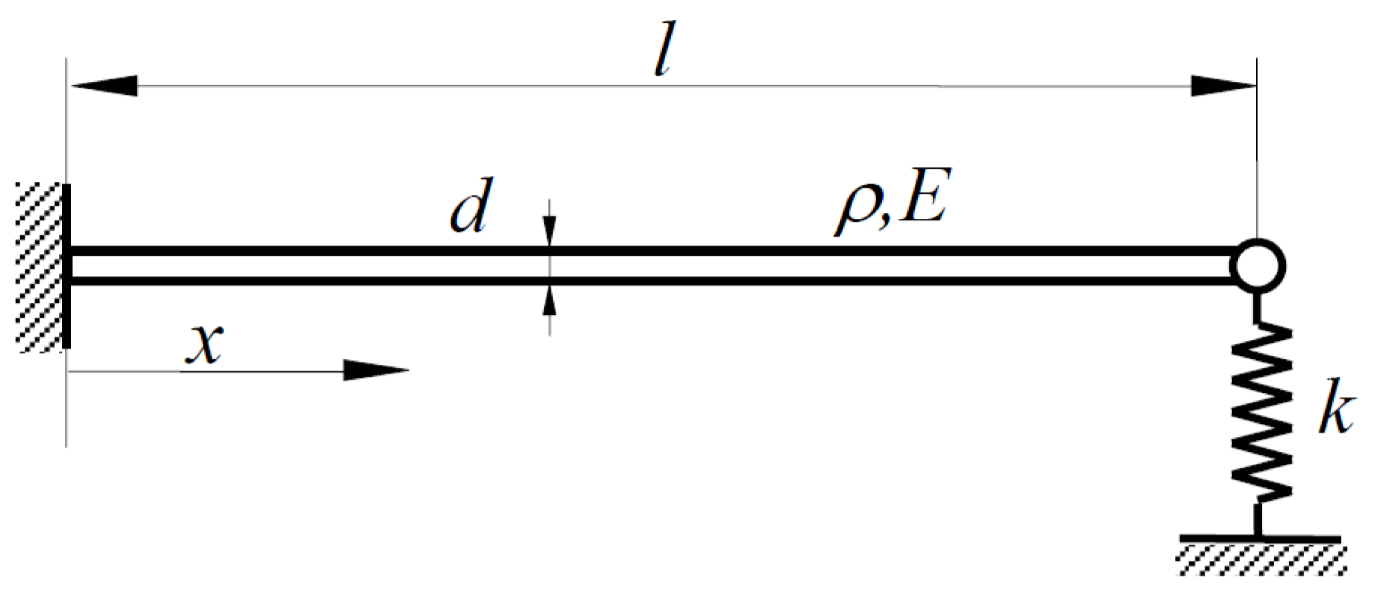## *Trave incastro - molla all'estremità opposta*

$Y(x) := A \cdot \cos(\beta \cdot x) + B \cdot \sin(\beta \cdot x) + C \cdot \cosh(\beta \cdot x) + D \cdot \sinh(\beta \cdot x)$

$Y'(x) := \dfrac{d}{dx} Y(x)$ $\qquad Y'(x) \text{ collect}, \beta \rightarrow (B \cdot \cos(\beta \cdot x) - A \cdot \sin(\beta \cdot x) + C \cdot \sinh(\beta \cdot x) + D \cdot \cosh(\beta \cdot x)) \cdot \beta$

$Y''(x) := \dfrac{d^2}{dx^2} Y(x)$ $\qquad Y''(x) \text{ collect}, \beta \rightarrow (C \cdot \cosh(\beta \cdot x) - B \cdot \sin(\beta \cdot x) - A \cdot \cos(\beta \cdot x) + D \cdot \sinh(\beta \cdot x)) \cdot \beta^2$

$Y'''(x) := \dfrac{d^3}{dx^3} Y(x)$ $\qquad Y'''(x) \text{ collect}, \beta \rightarrow (A \cdot \sin(\beta \cdot x) - B \cdot \cos(\beta \cdot x) + C \cdot \sinh(\beta \cdot x) + D \cdot \cosh(\beta \cdot x)) \cdot \beta^3$

$T(x) := E \cdot J \cdot \dfrac{d^3}{dx^3} Y(x)$ $\qquad$ *Azione di taglio*

Condizioni al contorno:

in x = 0

$Y(0) \rightarrow A + C$

$Y'(0) \text{ collect}, \beta \rightarrow (B + D) \cdot \beta$

in x = l

$Y''(l) \text{ collect}, \beta \rightarrow (C \cdot \cosh(\beta \cdot l) - B \cdot \sin(\beta \cdot l) - A \cdot \cos(\beta \cdot l) + D \cdot \sinh(\beta \cdot l)) \cdot \beta^2$

$T(l) - k \cdot Y(l) \text{ collect}, A, B, C, D \rightarrow \left(E \cdot J \cdot \beta^3 \cdot \sin(\beta \cdot l) - k \cdot \cos(\beta \cdot l)\right) \cdot A + \left(-k \cdot \sin(\beta \cdot l) - E \cdot J \cdot \beta^3 \cdot \cos(\beta \cdot l)\right) \cdot B + \left(E \cdot J \cdot \beta^3 \cdot \sinh(\beta \cdot l) - k \cdot \cosh(\beta \cdot l)\right) \cdot C + \left(E \cdot J \cdot \beta^3 \cdot \cosh(\beta \cdot l) - k \cdot \sinh(\beta \cdot l)\right) \cdot D$

Riassunto delle condizioni al contorno (che formano un sistema lineare)

$A + C = 0$

$B + D = 0$

$C \cdot \cosh(\beta \cdot l) - B \cdot \sin(\beta \cdot l) - A \cdot \cos(\beta \cdot l) + D \cdot \sinh(\beta \cdot l) = 0$

$\left(E \cdot J \cdot \beta^3 \cdot \sin(\beta \cdot l) - k \cdot \cos(\beta \cdot l)\right) \cdot A + \left(-k \cdot \sin(\beta \cdot l) - E \cdot J \cdot \beta^3 \cdot \cos(\beta \cdot l)\right) \cdot B + \left(E \cdot J \cdot \beta^3 \cdot \sinh(\beta \cdot l) - k \cdot \cosh(\beta \cdot l)\right) \cdot C + \left(E \cdot J \cdot \beta^3 \cdot \cosh(\beta \cdot l) - k \cdot \sinh(\beta \cdot l)\right) = 0$

Poniamo per semplicità: α = β l e otteniamo:

$A + C = 0$

$B + D = 0$

$C \cdot \cosh(\alpha) - B \cdot \sin(\alpha) - A \cdot \cos(\alpha) + D \cdot \sinh(\alpha) = 0$

$\left[E \cdot J \cdot \left(\dfrac{\alpha}{l}\right)^3 \cdot \sin(\alpha) - k \cdot \cos(\alpha)\right] \cdot A + \left[-k \cdot \sin(\alpha) - E \cdot J \cdot \left(\dfrac{\alpha}{l}\right)^3 \cdot \cos(\alpha)\right] \cdot B + \left[E \cdot J \cdot \left(\dfrac{\alpha}{l}\right)^3 \cdot \sinh(\alpha) - k \cdot \cosh(\alpha)\right] \cdot C + \left[E \cdot J \cdot \left(\dfrac{\alpha}{l}\right)^3 \cdot \cosh(\alpha) - k \cdot \sinh(\alpha)\right] = 0$

Quest'ultima condizione si può riscrivere nella forma:

$f_A(\alpha) \cdot A + f_B(\alpha) \cdot B + f_C(\alpha) \cdot C + f_D(\alpha) \cdot D = 0$

dove si è posto:

$h = \dfrac{E \cdot J}{l^3}$

$f_A(\alpha) := h \cdot \alpha^3 \cdot \sin(\alpha) - k \cdot \cos(\alpha)$

$f_B(\alpha) := -k \cdot \sin(\alpha) - h \cdot \alpha^3 \cdot \cos(\alpha)$

$f_C(\alpha) := h \cdot \alpha^3 \cdot \sinh(\alpha) - k \cdot \cosh(\alpha)$

$f_D(\alpha) := h \cdot \alpha^3 \cdot \cosh(\alpha) - k \cdot \sinh(\alpha)$

Le quattro condizioni al contorno si possono scrivere in forma matriciale, come segue

$$\mathbf{\Delta}(\alpha)\cdot\begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{\Delta}(\alpha) := \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -\cos(\alpha) & -\sin(\alpha) & \cosh(\alpha) & \sinh(\alpha) \\ f_A(\alpha) & f_B(\alpha) & f_C(\alpha) & f_D(\alpha) \end{pmatrix}$$

Conviene procedere per via numerica, senza sviluppi simbolici...

Lunghezza
$$l := 1$$

Materiale: acciaio

$$\rho := 7800$$

$$E := 206000\cdot10^6 = 2.06 \times 10^{11}$$

Sezione circolare di diametro d:

$$d := 20\cdot10^{-3} = 0.02$$

$$J := \frac{\pi\cdot d^4}{64} = 7.854 \times 10^{-9}$$

$$A_{sez} := \frac{\pi\cdot d^2}{4} = 3.142 \times 10^{-4}$$

Rigidezza della molla:

$$k := 20000\cdot1 = 2 \times 10^4$$

$$c := \sqrt{\frac{E\cdot J}{\rho\cdot A_{sez}}} = 25.695$$

$$h := \frac{E\cdot J}{l^3} = 1617.92$$

$$f_A(\alpha) := h\cdot\alpha^3\cdot\sin(\alpha) - k\cdot\cos(\alpha)$$

$$f_B(\alpha) := -k\cdot\sin(\alpha) - h\cdot\alpha^3\cdot\cos(\alpha)$$

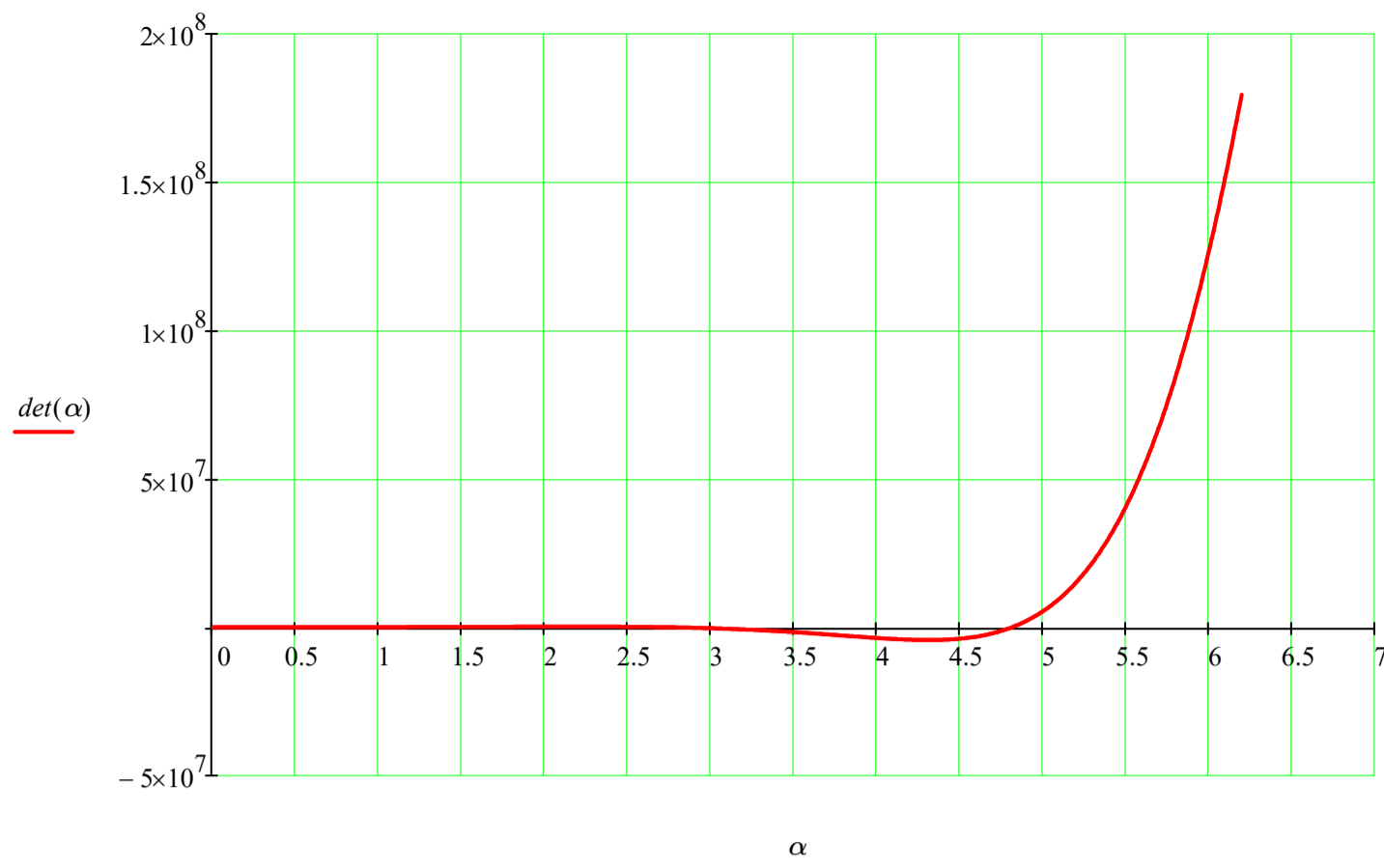$$f_C(\alpha) := h\cdot\alpha^3\cdot\sinh(\alpha) - k\cdot\cosh(\alpha)$$

$$f_D(\alpha) := h\cdot\alpha^3\cdot\cosh(\alpha) - k\cdot\sinh(\alpha)$$

$$\mathbf{\Delta}(\alpha) := \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -\cos(\alpha) & -\sin(\alpha) & \cosh(\alpha) & \sinh(\alpha) \\ f_A(\alpha) & f_B(\alpha) & f_C(\alpha) & f_D(\alpha) \end{pmatrix}$$

Cerchiamo le soluzioni dell'equazione caratteristica per via grafica:

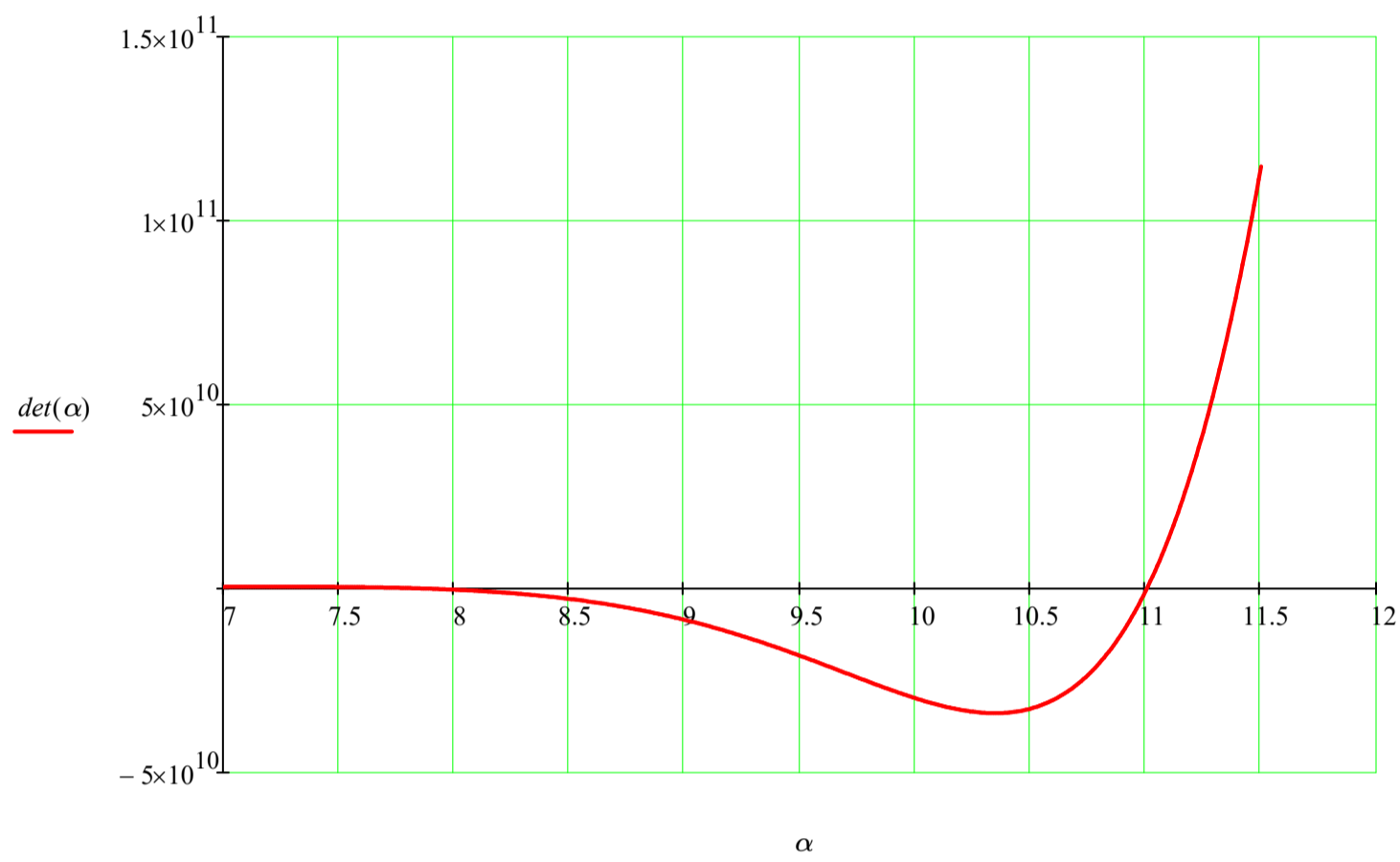$$det(\alpha) := |\mathbf{\Delta}(\alpha)|$$

$$\alpha := 0, 0.01 .. 6.2$$

$$\alpha := 3.5$$

$$\alpha_1 := root(det(\alpha), \alpha) = 2.735026$$

$$\alpha := 7$$

$$\alpha_2 := root(det(\alpha), \alpha) = 4.817982$$

$$\alpha := 7, 7.01 .. 11.5$$

$$\alpha := 10$$

$$\alpha_3 := root(det(\alpha), \alpha) = 7.880663$$

$$\alpha := 13$$

$$\alpha_4 := root(det(\alpha), \alpha) = 11.004902$$

$$\boxed{\alpha = \beta \cdot l} \qquad \boxed{\beta = \frac{\alpha}{l}}$$

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \begin{pmatrix} 2.735026 \\ 4.817982 \\ 7.880663 \\ 11.004902 \end{pmatrix} \qquad\qquad \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix} := \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} \cdot \frac{1}{l} = \begin{pmatrix} 2.735026 \\ 4.817982 \\ 7.880663 \\ 11.004902 \end{pmatrix}$$

$$\boxed{\omega = \beta^2 \cdot c}$$

$$\omega_1 := (\beta_1)^2 \cdot c = 192.211 \qquad\qquad f_1 := \frac{\omega_1}{2 \cdot \pi} = 30.591$$

$$\omega_2 := (\beta_2)^2 \cdot c = 596.467 \qquad\qquad f_2 := \frac{\omega_2}{2 \cdot \pi} = 94.931$$

$$\omega_3 := (\beta_3)^2 \cdot c = 1595.812 \qquad\qquad f_3 := \frac{\omega_3}{2 \cdot \pi} = 253.981$$

$$\omega_4 := (\beta_4)^2 \cdot c = 3111.922 \qquad\qquad f_4 := \frac{\omega_4}{2 \cdot \pi} = 495.278$$

$$l := l$$

Given

$$\boxed{A + C = 0}$$

$$\boxed{B + D = 0}$$

$$\boxed{C \cdot \cosh(\beta \cdot l) - B \cdot \sin(\beta \cdot l) - A \cdot \cos(\beta \cdot l) + D \cdot \sinh(\beta \cdot l) = 0}$$

$$Find(B, C, D) \rightarrow \begin{pmatrix} -\dfrac{A \cdot \cos(\beta \cdot l) + A \cdot \cosh(\beta \cdot l)}{\sinh(\beta \cdot l) + \sin(\beta \cdot l)} \\ -A \\ \dfrac{A \cdot \cos(\beta \cdot l) + A \cdot \cosh(\beta \cdot l)}{\sinh(\beta \cdot l) + \sin(\beta \cdot l)} \end{pmatrix}$$

$A := 1$

$$B(\beta) := -\frac{A \cdot cos(\beta \cdot l) + A \cdot cosh(\beta \cdot l)}{sinh(\beta \cdot l) + sin(\beta \cdot l)}$$
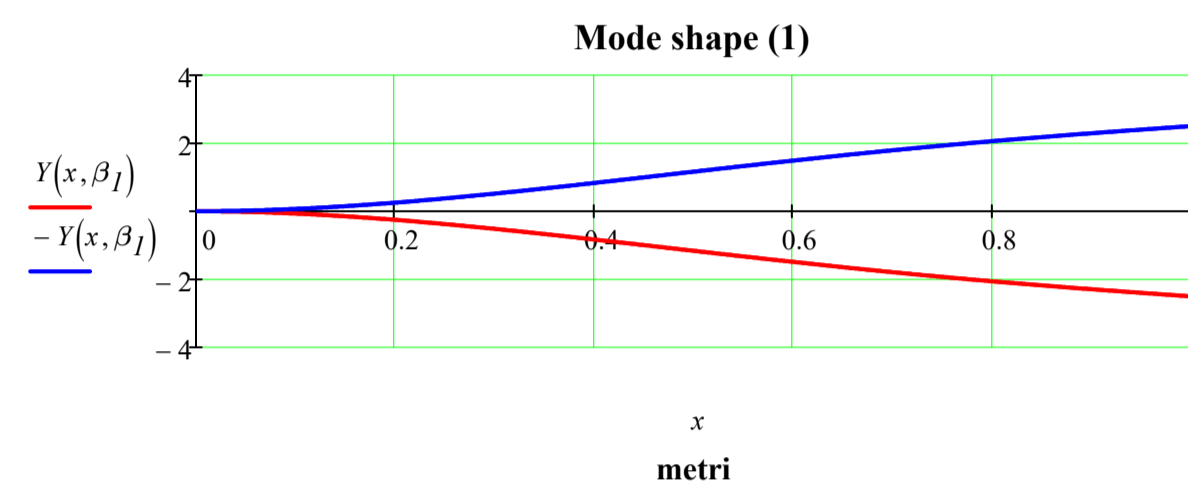
$C := -A$

$$D(\beta) := \frac{A \cdot cos(\beta \cdot l) + A \cdot cosh(\beta \cdot l)}{sinh(\beta \cdot l) + sin(\beta \cdot l)}$$

$Y(x, \beta) := A \cdot cos(\beta \cdot x) + B(\beta) \cdot sin(\beta \cdot x) + C \cdot cosh(\beta \cdot x) + D(\beta) \cdot sinh(\beta \cdot x)$
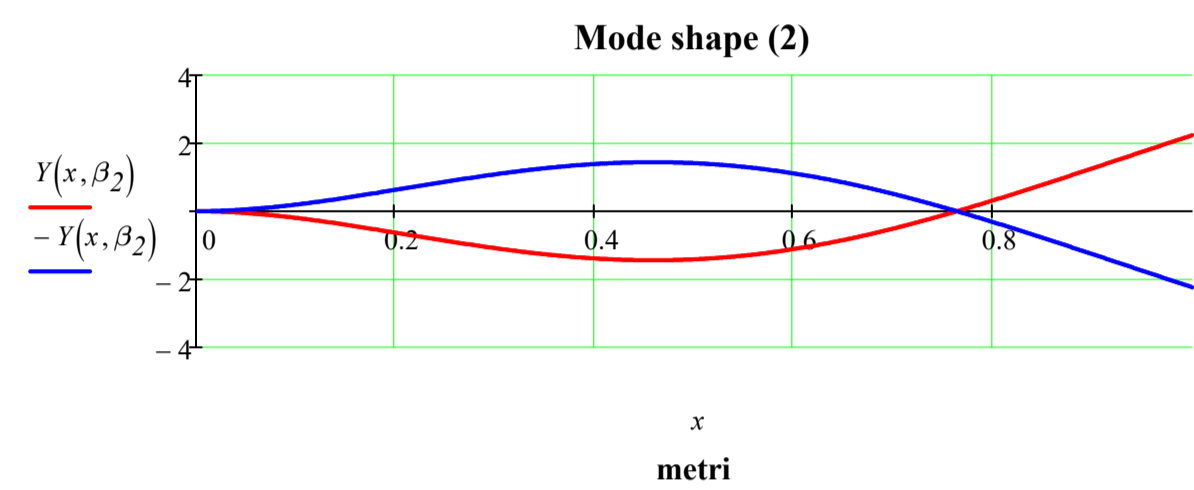
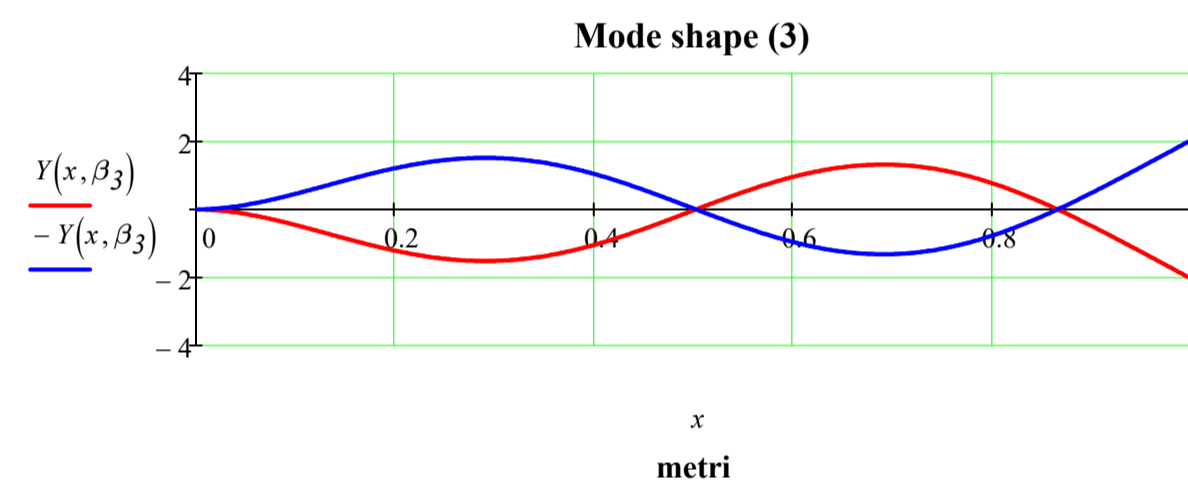$x := 0, \dfrac{l}{500} .. l$

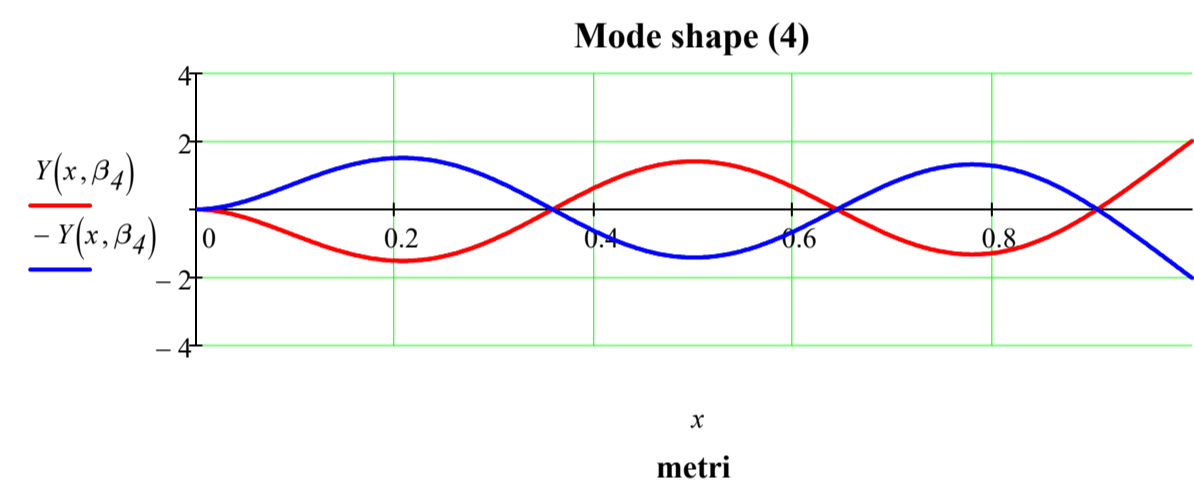$FS := 4$ *Fondo scala per i grafici*

$f_1 = 30.591 \, s \cdot Hz$

Mode shape (1)

$f_2 = 94.931 \, s \cdot Hz$

Mode shape (2)

$f_3 = 253.981 \, s \cdot Hz$

Mode shape (3)

$f_4 = 495.278 \, s \cdot Hz$

Mode shape (4)

Animazioni delle deformate (Usare il menù: Strumenti - Aminazione - Registra)

Mode shape (1)

Mode shape (2)

Mode shape (3)

Mode shape (4)