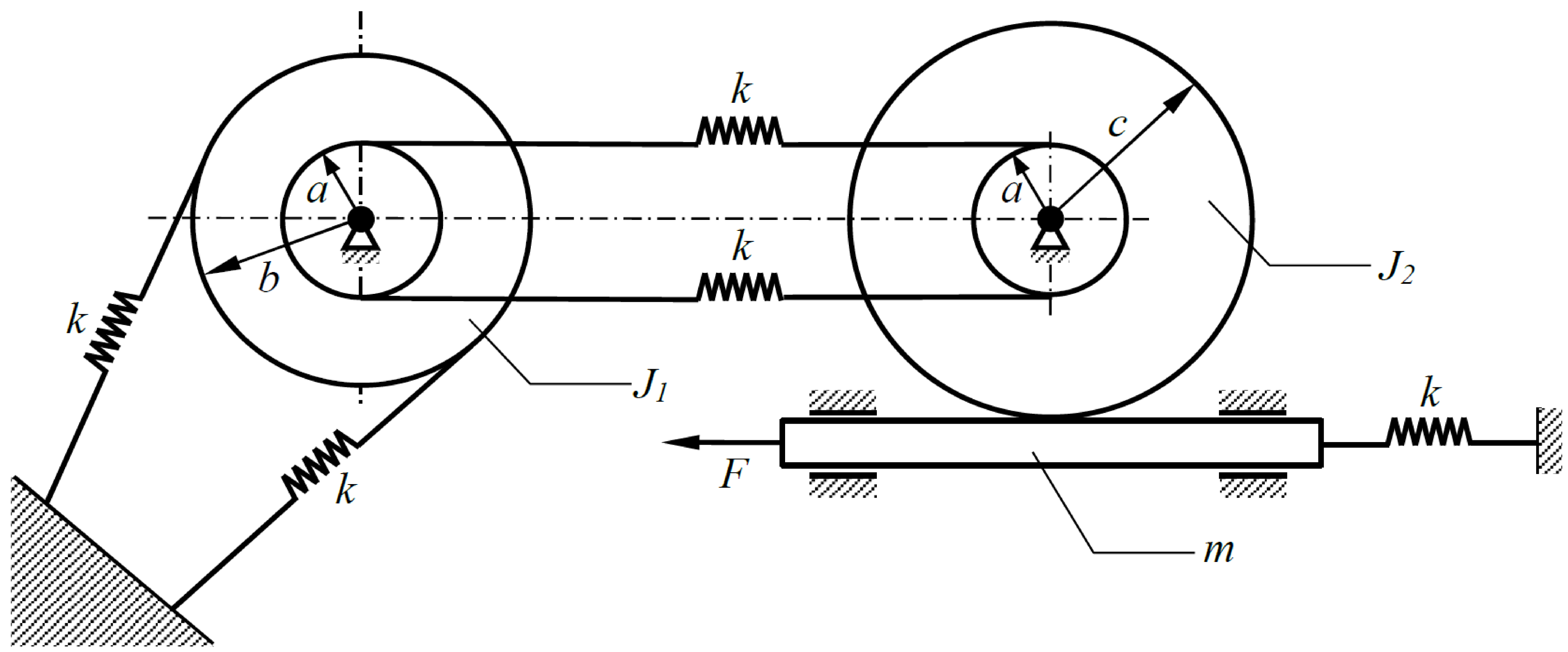
Uso delle coordinate principali: sistema a due gdl smorzato (con smorzamento proporzionale)



Parametri geometrici

$a := 80 \cdot 10^{-3} = 0.08$

$b := 180 \cdot 10^{-3} = 0.18$

$c := 220 \cdot 10^{-3} = 0.22$

Parametri inerziali

$J_1 := 1.2$

$J_2 := 3$

$m := 15$

Parametri elastici

$k := 2500$

$V(\alpha, \beta) := \frac{1}{2} \cdot \left[2 \cdot k \cdot b^2 \cdot \alpha^2 + 2 \cdot k \cdot (a \cdot \alpha - a \cdot \beta)^2 + k \cdot c^2 \cdot \beta^2\right]$ Energia potenziale

$\frac{\partial}{\partial \alpha} V(\alpha, \beta) \rightarrow 162.0 \cdot \alpha + -32.0 \cdot \beta + 32.0 \cdot \alpha$

$\frac{\partial}{\partial \beta} V(\alpha, \beta) \rightarrow 121.0 \cdot \beta + 32.0 \cdot \beta + -32.0 \cdot \alpha$

Matrici di massa e di rigidezza

$$\mathbf{J} := \begin{bmatrix} J_1 & 0 \\ 0 & \left(J_2 + m \cdot c^2\right) \end{bmatrix} = \begin{pmatrix} 1.2 & 0 \\ 0 & 3.726 \end{pmatrix} \qquad diag\left(\begin{bmatrix} J_1 \\ \left(J_2 + m \cdot c^2\right) \end{bmatrix}\right) = \begin{pmatrix} 1.2 & 0 \\ 0 & 3.726 \end{pmatrix}$$

$$\mathbf{K} := \begin{bmatrix} 2 \cdot k \cdot \left(a^2 + b^2\right) & -2 \cdot k \cdot a^2 \\ -2 \cdot k \cdot a^2 & k \cdot \left(2 \cdot a^2 + c^2\right) \end{bmatrix} = \begin{pmatrix} 194 & -32 \\ -32 & 153 \end{pmatrix}$$

$\alpha_R := 0.02$ $\beta_R := 0.03$

Coefficienti di Rayleigh

$$\mathbf{C} := \alpha_R \cdot \mathbf{J} + \beta_R \cdot \mathbf{K} = \begin{pmatrix} 5.844 & -0.96 \\ -0.96 & 4.665 \end{pmatrix}$$

Matrice di smorzamento, calcolata con l'ipotesi di "smorzamento proporzionale" (o smorzamento di Rayleigh)

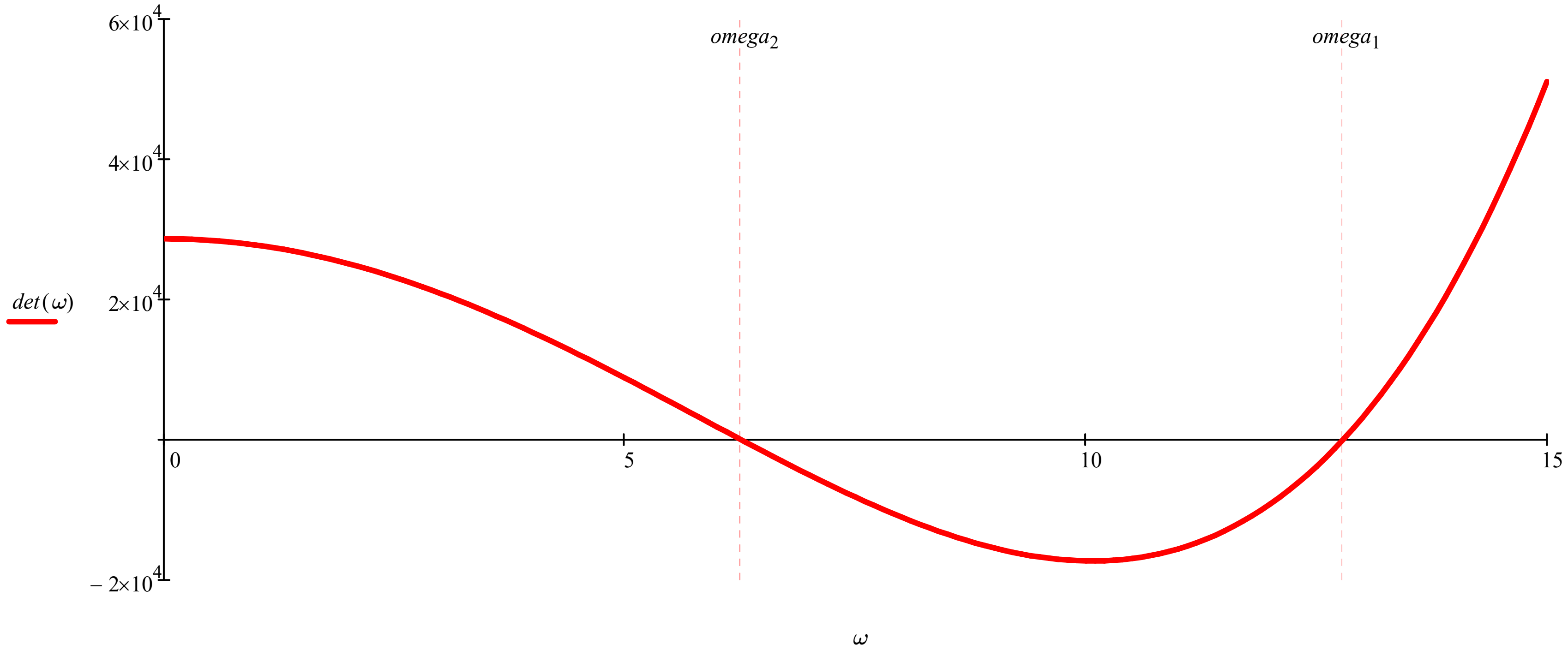
Calcolo delle pulsazioni proprie

$$omega := \sqrt{genvals(\mathbf{K}, \mathbf{J})} = \begin{pmatrix} 12.788 \\ 6.26 \end{pmatrix} \qquad \sqrt{eigenvals\left(\mathbf{J}^{-1} \cdot \mathbf{K}\right)} = \begin{pmatrix} 12.788 \\ 6.26 \end{pmatrix}$$

$$\Delta(\omega) := \mathbf{K} - \omega^2 \cdot \mathbf{J}$$

$$det(\omega) := \left|\Delta(\omega)\right|$$

$\omega := 0, 0.1 .. 15$



$$\omega := \sqrt{genvals(\mathbf{K},\mathbf{J})} = \begin{pmatrix} 12.788 \\ 6.26 \end{pmatrix}$$

$$\mathbf{\Phi} := genvecs(\mathbf{K},\mathbf{J}) = \begin{pmatrix} 1 & 0.218 \\ -0.07 & 1 \end{pmatrix}$$

$$\mathbf{X_1} := \mathbf{\Phi}^{\langle 1 \rangle} = \begin{pmatrix} 1 \\ -0.07 \end{pmatrix} \qquad \omega_1 = 12.788$$

$$\mathbf{X_2} := \mathbf{\Phi}^{\langle 2 \rangle} = \begin{pmatrix} 0.218 \\ 1 \end{pmatrix} \qquad \omega_2 = 6.26$$

$$\mathbf{J_{star}} := \mathbf{\Phi}^T \cdot \mathbf{J} \cdot \mathbf{\Phi} = \begin{pmatrix} 1.218 & 0 \\ 0 & 3.783 \end{pmatrix}$$

$$\mathbf{K_{star}} := \mathbf{\Phi}^T \cdot \mathbf{K} \cdot \mathbf{\Phi} = \begin{pmatrix} 199.24 & 0 \\ 0 & 148.262 \end{pmatrix}$$

$$\mathbf{C_{star}} := \alpha_R \cdot \mathbf{J_{star}} + \beta_R \cdot \mathbf{K_{star}} = \begin{pmatrix} 6.002 & 0 \\ 0 & 4.524 \end{pmatrix}$$

$$\omega_1 = 12.788 \qquad \sqrt{\frac{\mathbf{K_{star}}_{1,1}}{\mathbf{J_{star}}_{1,1}}} = 12.788$$

$$\omega_2 = 6.26 \qquad \sqrt{\frac{\mathbf{K_{star}}_{2,2}}{\mathbf{J_{star}}_{2,2}}} = 6.26$$

$$\mu_1 := \frac{1}{\sqrt{\mathbf{X_1}^T \cdot \mathbf{J} \cdot \mathbf{X_1}}} = 0.906 \qquad \mu_2 := \frac{1}{\sqrt{\mathbf{X_2}^T \cdot \mathbf{J} \cdot \mathbf{X_2}}} = 0.514$$

$$\mathbf{X_1} = \begin{pmatrix} 1 \\ -0.07 \end{pmatrix} \qquad \mathbf{X_2} = \begin{pmatrix} 0.218 \\ 1 \end{pmatrix}$$

$$\mathbf{X_1} := \mu_1 \cdot \mathbf{X_1} = \begin{pmatrix} 0.906 \\ -0.064 \end{pmatrix} \qquad \mathbf{X_2} := \mu_2 \cdot \mathbf{X_2} = \begin{pmatrix} 0.112 \\ 0.514 \end{pmatrix}$$

$$\mathbf{\Phi} = \begin{pmatrix} 1 & 0.218 \\ -0.07 & 1 \end{pmatrix}$$

$$\mathbf{\Phi} := augment(\mathbf{X_1},\mathbf{X_2}) = \begin{pmatrix} 0.906 & 0.112 \\ -0.064 & 0.514 \end{pmatrix}$$

$$\mathbf{\Phi}^T \cdot \mathbf{J} \cdot \mathbf{\Phi} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{W} := \mathbf{\Phi}^T \cdot \mathbf{K} \cdot \mathbf{\Phi} = \begin{pmatrix} 163.537 & 0 \\ 0 & 39.193 \end{pmatrix} \qquad (\omega_1)^2 = 163.537 \qquad (\omega_2)^2 = 39.193$$

Forzante

$$F := 500$$

$$\mathbf{F} := \begin{pmatrix} 0 \\ F \cdot c \end{pmatrix} \qquad \mathbf{F} = \begin{pmatrix} 0 \\ 110 \end{pmatrix}$$

$$\mathbf{\Phi}^T = \begin{pmatrix} 0.906 & -0.064 \\ 0.112 & 0.514 \end{pmatrix}$$

$$\mathbf{Q} := \mathbf{\Phi}^T \cdot \mathbf{F} = \begin{pmatrix} -6.988 \\ 56.556 \end{pmatrix}$$

Equazioni di moto in formato matriciale (nelle coord. principali p_1 e p_2)

$$\begin{pmatrix} p''_1(t) \\ p''_2(t) \end{pmatrix} + \left[\alpha_R \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \beta_R \cdot \begin{bmatrix} (\omega_1)^2 & 0 \\ 0 & (\omega_2)^2 \end{bmatrix} \right] \cdot \begin{pmatrix} p'_1(t) \\ p'_2(t) \end{pmatrix} + \begin{bmatrix} (\omega_1)^2 & 0 \\ 0 & (\omega_2)^2 \end{bmatrix} \cdot \begin{pmatrix} p_1(t) \\ p_2(t) \end{pmatrix} = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}$$

$$\alpha_R + (\omega_1)^2 \cdot \beta_R = 4.926 \qquad (\omega_1)^2 = 163.537 \qquad \mathbf{Q}_1 = -6.988$$

$$\alpha_R + (\omega_2)^2 \cdot \beta_R = 1.196 \qquad (\omega_2)^2 = 39.193 \qquad \mathbf{Q}_2 = 56.556$$

$$\mathbf{p}(t) = \begin{pmatrix} p_1(t) \\ p_2(t) \end{pmatrix}$$

Calcolo dei fattori di smorzamento modali e delle corrispondenti pulsazioni proprie smorzate

$$\xi_1 := \frac{\alpha_R + (\omega_1)^2 \cdot \beta_R}{2 \cdot \omega_1} = 0.193 \qquad \omega_{s1} := \omega_1 \cdot \sqrt{1 - {\xi_1}^2} = 12.549 \qquad \omega_1 = 12.788$$

$$\xi_2 := \frac{\alpha_R + (\omega_2)^2 \cdot \beta_R}{2 \cdot \omega_2} = 0.096 \qquad \omega_{s2} := \omega_2 \cdot \sqrt{1 - {\xi_2}^2} = 6.232 \qquad \omega_2 = 6.26$$

Soluzione delle equazioni in coord. principali (dalla teoria dei sistemi a 1 gdl smorzati):

$$p_1(t) = e^{-\xi_1 \cdot \omega_1 \cdot t} \cdot \left(A_1 \cdot \cos(\omega_{s1} \cdot t) + B_1 \cdot \sin(\omega_{s1} \cdot t)\right) + \frac{\mathbf{Q}_1}{(\omega_1)^2}$$

$$p_2(t) = e^{-\xi_2 \cdot \omega_2 \cdot t} \cdot \left(A_2 \cdot \cos(\omega_{s2} \cdot t) + B_2 \cdot \sin(\omega_{s2} \cdot t)\right) + \frac{\mathbf{Q}_2}{(\omega_2)^2}$$

Derivate temporali prime (velocità):

$$p'_1(t) = \left[-e^{-\xi_1 \cdot \omega_1 \cdot t} \cdot \left(A_1 \cdot \omega_{s1} + B_1 \cdot \xi_1 \cdot \omega_1\right)\right] \cdot \sin(\omega_{s1} \cdot t) + e^{-\xi_1 \cdot \omega_1 \cdot t} \cdot \left(B_1 \cdot \omega_{s1} - A_1 \cdot \xi_1 \cdot \omega_1\right) \cdot \cos(\omega_{s1} \cdot t)$$

$$p'_2(t) = \left[-e^{-\xi_2 \cdot \omega_2 \cdot t} \cdot \left(A_2 \cdot \omega_{s2} + B_2 \cdot \xi_2 \cdot \omega_2\right)\right] \cdot \sin(\omega_{s2} \cdot t) + e^{-\xi_2 \cdot \omega_2 \cdot t} \cdot \left(B_2 \cdot \omega_{s2} - A_2 \cdot \xi_2 \cdot \omega_2\right) \cdot \cos(\omega_{s2} \cdot t)$$

Occorre assegnare le condizioni iniziali per calcolare le costanti A_1, B_1, A_2, B_2

Le condizioni iniziali vengono sempre assegnate nelle coordinate fisiche; occorre quindi convertirle nelle coordinate principali.

Si utilizzano le seguenti trasformazioni:

$$\mathbf{p}(t) = \Phi^{-1} \cdot \mathbf{x}(t)$$ per le posizioni

$$\mathbf{p'}(t) = \Phi^{-1} \cdot \mathbf{x'}(t)$$ per le velocità

All'istante iniziale t=0 si avrà:

$$\mathbf{p}(0) = \Phi^{-1} \cdot \mathbf{x}(0)$$

$$\mathbf{p'}(0) = \Phi^{-1} \cdot \mathbf{x'}(0)$$

Assegniamo le condizioni iniziali nelle coordinate fisiche

$$\mathbf{x}(0) = \begin{pmatrix} \alpha_0 \\ \beta_0 \end{pmatrix} \qquad \mathbf{x'}(0) = \begin{pmatrix} \alpha'_0 \\ \beta'_0 \end{pmatrix}$$

$\alpha_0 := 20 \cdot deg = 0.349 \cdot rad$ $\beta_0 := 10 \cdot deg = 0.175 \cdot rad$

$\alpha'_0 := 2$ $\beta'_0 := -3$

Convertiamo le condizioni iniziali nelle coordinate principali

Per le posizioni:

$$\mathbf{p}_{(0)} = \mathbf{\Phi}^{-1} \cdot \mathbf{x}_{(0)}$$

$$\begin{pmatrix} p_{10} \\ p_{20} \end{pmatrix} := \mathbf{\Phi}^{-1} \cdot \begin{pmatrix} \alpha_0 \\ \beta_0 \end{pmatrix} = \begin{pmatrix} 0.338 \\ 0.381 \end{pmatrix}$$

$p_{10} = 0.338$ $p_{20} = 0.381$

Per le velocità:

$$\mathbf{p'}_{(0)} = \mathbf{\Phi}^{-1} \cdot \mathbf{x'}_{(0)}$$

$$\begin{pmatrix} p'_{10} \\ p'_{20} \end{pmatrix} := \mathbf{\Phi}^{-1} \cdot \begin{pmatrix} \alpha'_0 \\ \beta'_0 \end{pmatrix} = \begin{pmatrix} 2.884 \\ -5.478 \end{pmatrix}$$

$p'_{10} = 2.884$ $p'_{20} = -5.478$

Imponendo le condizioni iniziali si ha:

$$\begin{pmatrix} A_1 \\ B_1 \\ A_2 \\ B_2 \end{pmatrix} := \begin{bmatrix} \dfrac{p_{10} \cdot (\omega_1)^2 - \mathbf{Q}_1}{(\omega_1)^2} \\ \dfrac{p_{10} \cdot \xi_1 \cdot (\omega_1)^2 + p'_{10} \cdot \omega_1 - \xi_1 \cdot \mathbf{Q}_1}{\omega_{s1} \cdot \omega_1} \\ \dfrac{p_{20} \cdot (\omega_2)^2 - \mathbf{Q}_2}{(\omega_2)^2} \\ \dfrac{p_{20} \cdot \xi_2 \cdot (\omega_2)^2 + p'_{20} \cdot \omega_2 - \xi_2 \cdot \mathbf{Q}_2}{\omega_{s2} \cdot \omega_2} \end{bmatrix} = \begin{pmatrix} 0.381 \\ 0.305 \\ -1.062 \\ -0.981 \end{pmatrix}$$

La soluzione nelle coordinate principali è ora calcolabile, perché sono state determinate le costanti A_1, B_1, A_2, B_2

$$p_1(t) := e^{-\xi_1 \cdot \omega_1 \cdot t} \cdot \left(A_1 \cdot cos\left(\omega_{s1} \cdot t\right) + B_1 \cdot sin\left(\omega_{s1} \cdot t\right)\right) + \frac{\mathbf{Q}_1}{\left(\omega_1\right)^2}$$

$$p_2(t) := e^{-\xi_2 \cdot \omega_2 \cdot t} \cdot \left(A_2 \cdot cos\left(\omega_{s2} \cdot t\right) + B_2 \cdot sin\left(\omega_{s2} \cdot t\right)\right) + \frac{\mathbf{Q}_2}{\left(\omega_2\right)^2}$$

Occorre infine applicare la formula 5.195 per calcolare la soluzione nelle coordinate fisiche α e β

$$\mathbf{p}(t) := \begin{pmatrix} p_1(t) \\ p_2(t) \end{pmatrix}$$

$$\mathbf{x}(t) := \mathbf{\Phi} \cdot \mathbf{p}(t)$$

$$\alpha(t) := \mathbf{x}(t)_1$$

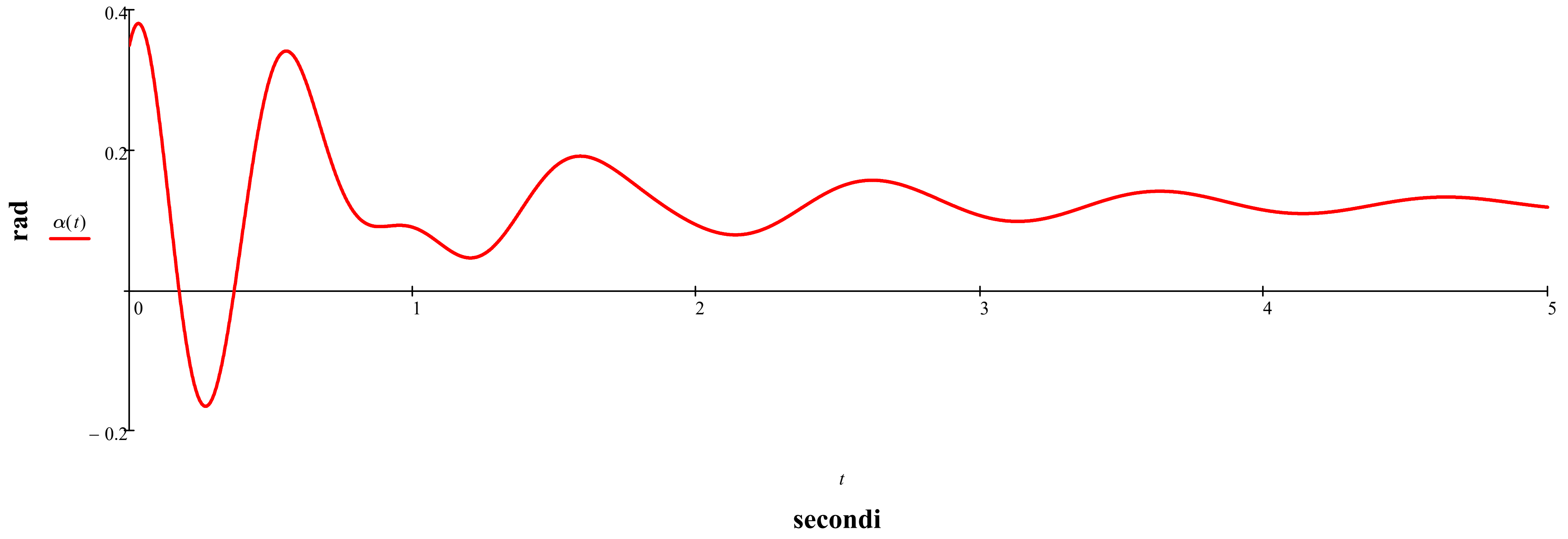
$$\beta(t) := \mathbf{x}(t)_2$$

$T_{max} := 9$

$\Delta t := 0.001$

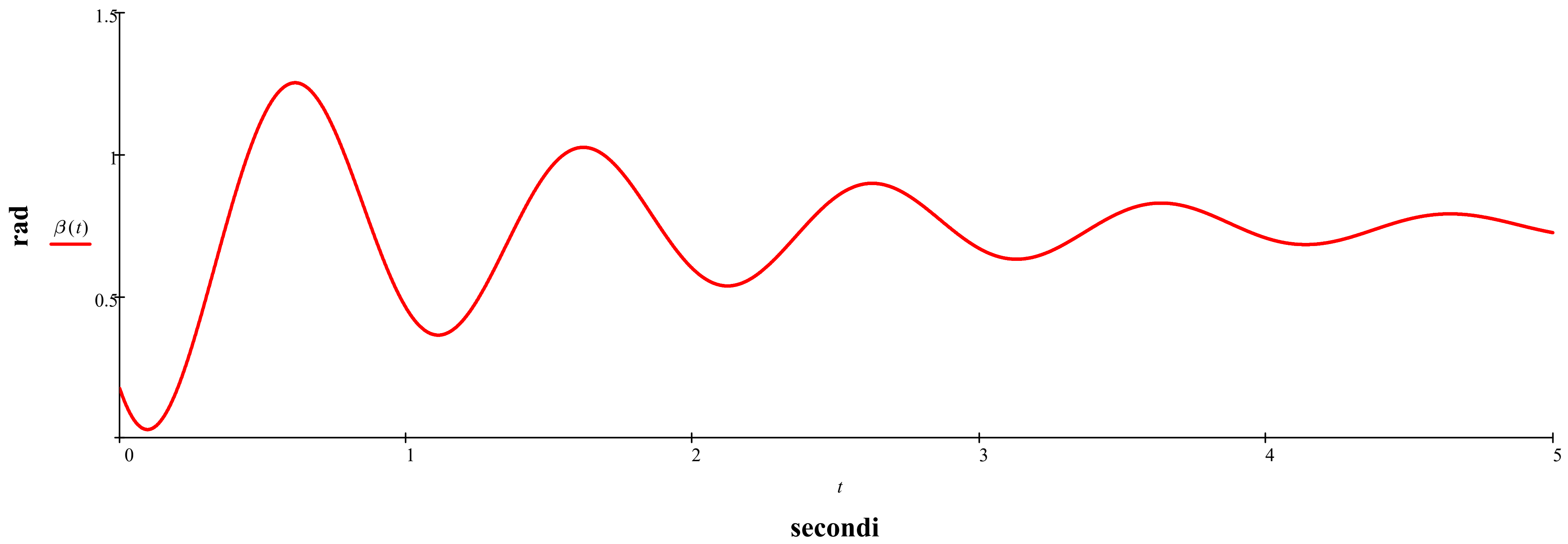
$T_{max} := 5$

$t := 0, \Delta t .. T_{max}$



$\alpha_R \equiv 0.1$ $\beta_R \equiv 0.1$

$$\mathbf{C} = \begin{pmatrix} 5.844 & -0.96 \\ -0.96 & 4.665 \end{pmatrix}$$



Verifica dei calcoli mediante integrazione numerica

$$\mathbf{I} := identity(2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \mathbf{O} := 0 \cdot \mathbf{I} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\mathbf{A_{sup}} := augment(\mathbf{O}, \mathbf{I}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{A_{inf}} := augment\left(-\mathbf{J}^{-1} \cdot \mathbf{K}, -\mathbf{J}^{-1} \cdot \mathbf{C}\right) = \begin{pmatrix} -161.667 & 26.667 & -4.87 & 0.8 \\ 8.588 & -41.063 & 0.258 & -1.252 \end{pmatrix}$$

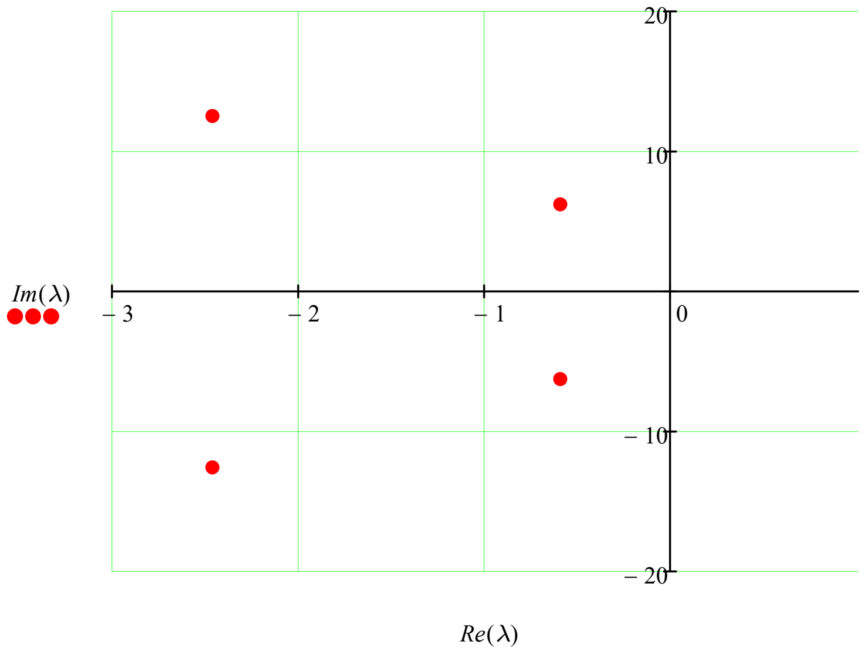
$$\mathbf{A} := stack\left(\mathbf{A_{sup}}, \mathbf{A_{inf}}\right) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -161.667 & 26.667 & -4.87 & 0.8 \\ 8.588 & -41.063 & 0.258 & -1.252 \end{pmatrix}$$

$$\mathbf{o} := \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\mathbf{F} = \begin{pmatrix} 0 \\ 110 \end{pmatrix}$$

$$\mathbf{b} := stack\left(\mathbf{o}, \mathbf{J}^{-1} \cdot \mathbf{F}\right) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 29.522 \end{pmatrix}$$

$$\lambda := eigenvals(\mathbf{A}) = \begin{pmatrix} -2.463 + 12.549i \\ -2.463 - 12.549i \\ -0.598 + 6.232i \\ -0.598 - 6.232i \end{pmatrix} \qquad \omega = \begin{pmatrix} 12.788 \\ 6.26 \end{pmatrix}$$



$$\mathbf{y} := \begin{pmatrix} \alpha_0 \\ \beta_0 \\ \alpha'_0 \\ \beta'_0 \end{pmatrix} = \begin{pmatrix} 0.349 \\ 0.175 \\ 2 \\ -3 \end{pmatrix}$$

Condizioni iniziali

$$\Delta t = 1 \times 10^{-3}$$

$$N := ceil\left(\frac{T_{max}}{\Delta t}\right) = 5000$$

$$EQMOTO(t, \mathbf{y}) := \mathbf{A} \cdot \mathbf{y} + \mathbf{b}$$

Equazioni di moto nello spazio di stato

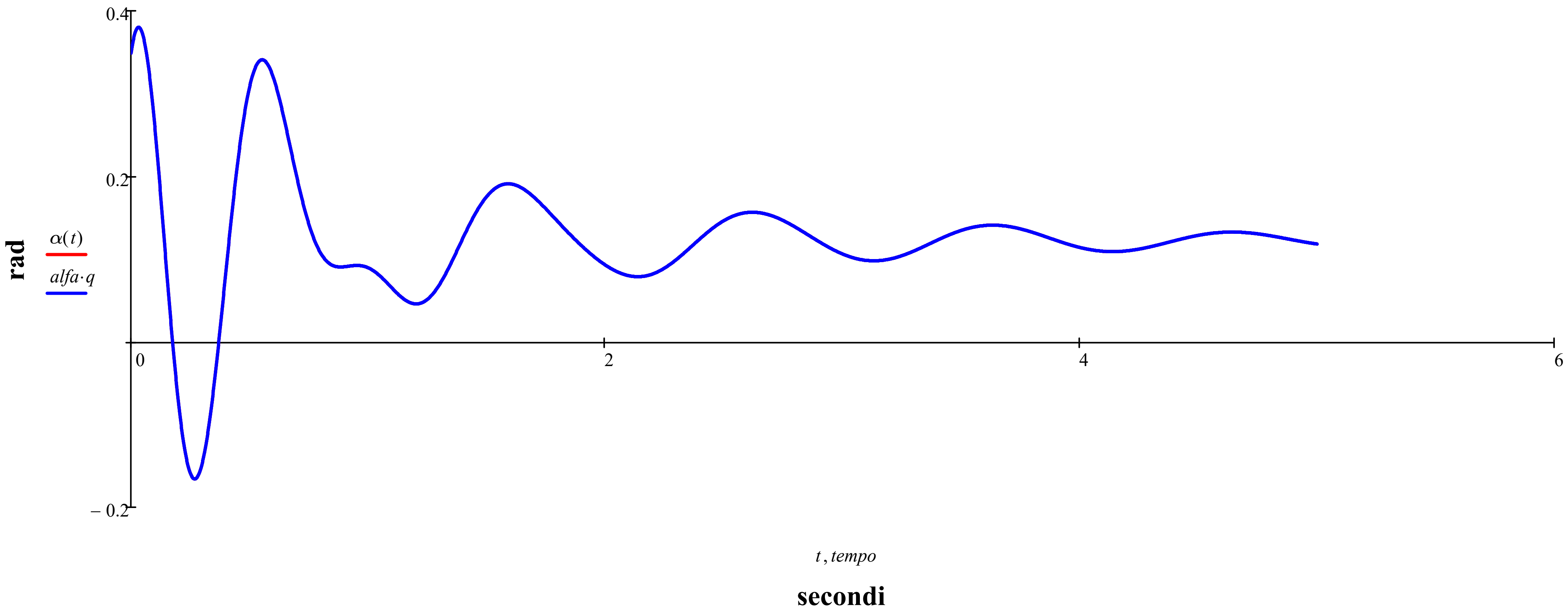
$$TAB := rkfixed\left(\mathbf{y}, 0, T_{max}, N, EQMOTO\right)$$

Soluzione per via numerica (Runge-Kutta)

$tempo := TAB^{\langle 1 \rangle}$

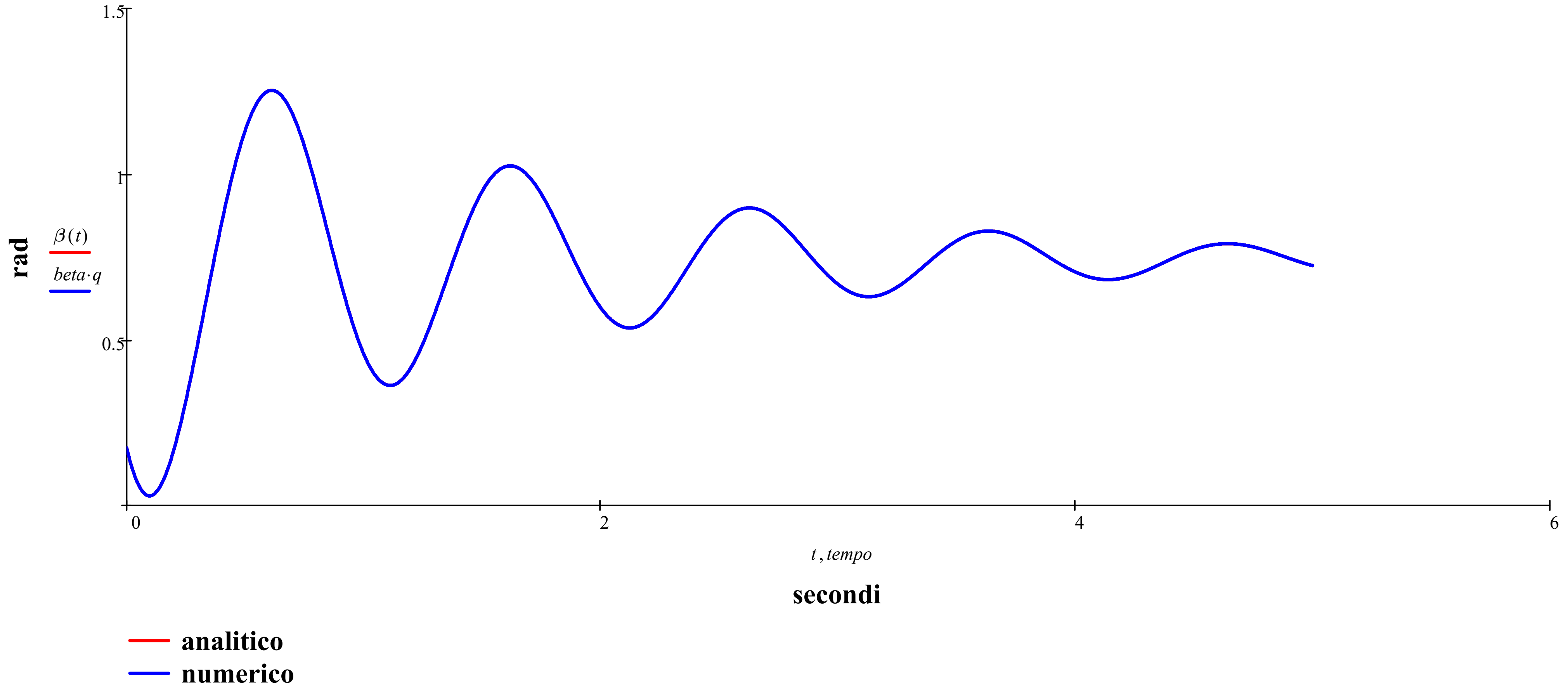
Se il procedimento è corretto le due tracce (metodo analitico e numerico) devono sempre sovrapporsi

$alfa := TAB^{\langle 2 \rangle}$ $beta := TAB^{\langle 3 \rangle}$ $q := 1$ Variabile on/off



analitico

numerico



1.5
1
0.5
rad
β(t)
beta·q
0
2
4
6
t , tempo
secondi
analitico
numerico