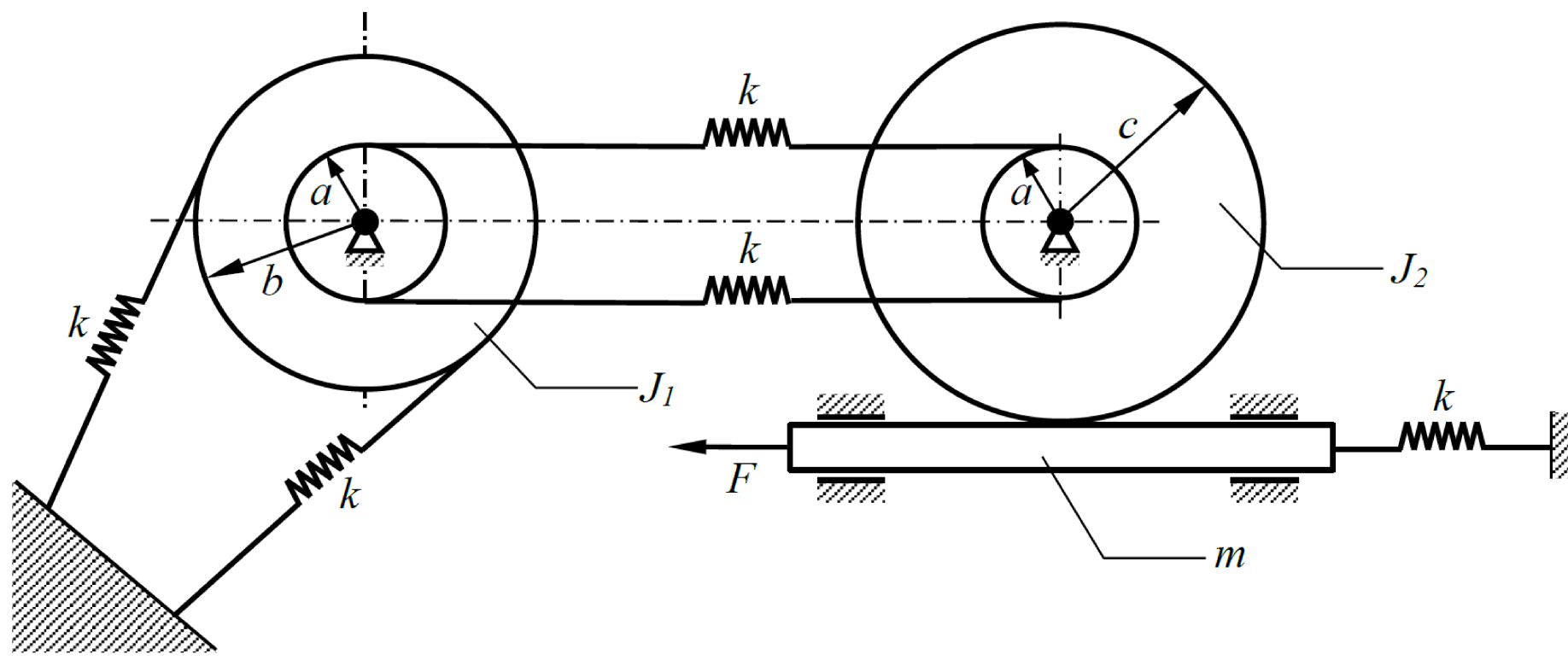


## Uso delle coordinate principali: sistema a due gdl non smorzato



### Parametri geometrici

$$a := 80 \cdot 10^{-3} = 0.08$$

$$b := 180 \cdot 10^{-3} = 0.18$$

$$c := 220 \cdot 10^{-3} = 0.22$$

### Parametri inerziali

$$J_1 := 1.2$$

$$J_2 := 3$$

$$m := 15$$

### Parametri elastici

$$k := 2500$$

$$V(\alpha, \beta) := \frac{1}{2} \cdot [2 \cdot k \cdot b^2 \cdot \alpha^2 + 2 \cdot k \cdot (a \cdot \alpha - a \cdot \beta)^2 + k \cdot c^2 \cdot \beta^2]$$

## Energia potenziale

$$\frac{\partial}{\partial \alpha} V(\alpha, \beta) \text{ simplify} \rightarrow 194.0 \cdot \alpha - 32.0 \cdot \beta$$

$$\frac{\partial}{\partial \beta} V(\alpha, \beta) \text{ simplify} \rightarrow 153.0 \cdot \beta - 32.0 \cdot \alpha$$

## Matrici di massa e di rigidezza

$$\mathbf{J} := \begin{bmatrix} J_1 & 0 \\ 0 & (J_2 + m \cdot c^2) \end{bmatrix} = \begin{pmatrix} 1.2 & 0 \\ 0 & 3.726 \end{pmatrix}$$

$$\text{diag} \left[ \begin{bmatrix} J_1 \\ (J_2 + m \cdot c^2) \end{bmatrix} \right] = \begin{pmatrix} 1.2 & 0 \\ 0 & 3.726 \end{pmatrix}$$

$$\mathbf{K} := \begin{bmatrix} 2 \cdot k \cdot (a^2 + b^2) & -2 \cdot k \cdot a^2 \\ -2 \cdot k \cdot a^2 & k \cdot (2 \cdot a^2 + c^2) \end{bmatrix} = \begin{pmatrix} 194 & -32 \\ -32 & 153 \end{pmatrix}$$

## Calcolo delle pulsazioni proprie

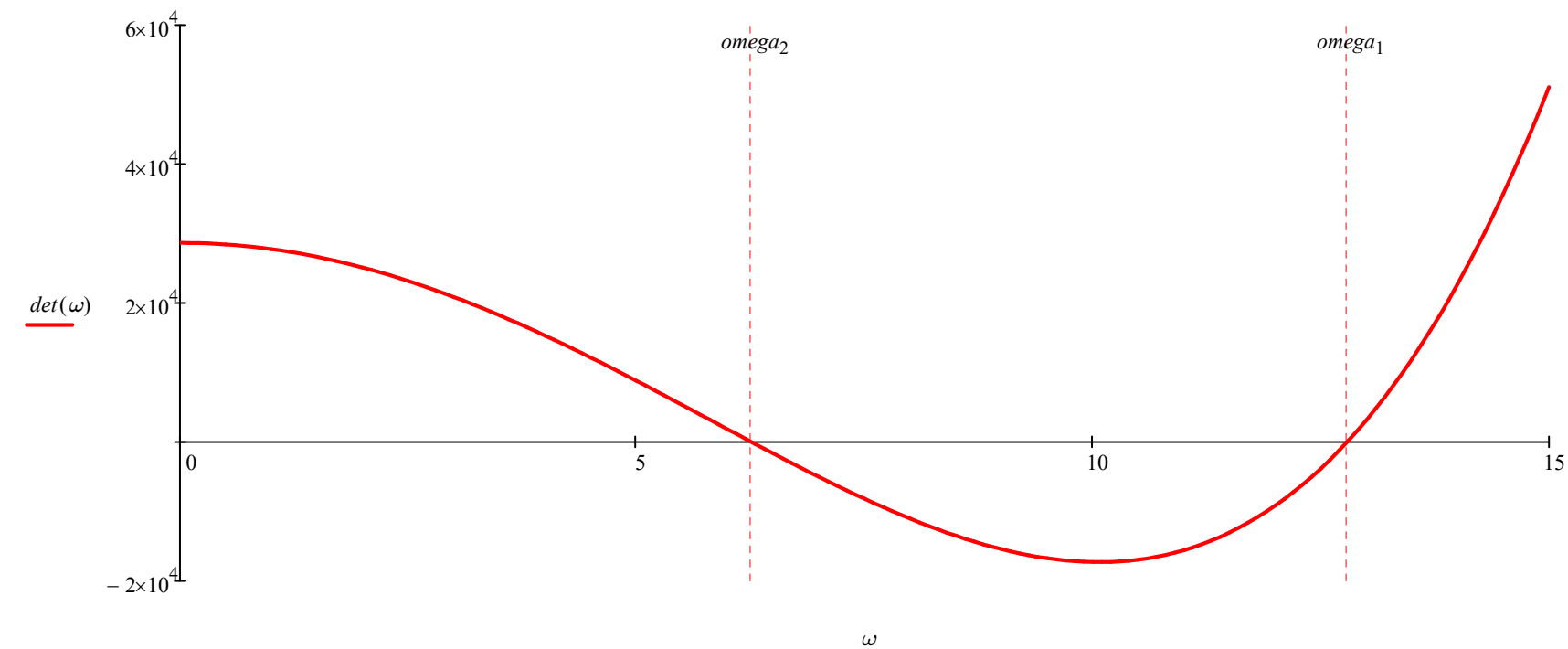
$$\omega := \sqrt{\text{genvals}(\mathbf{K}, \mathbf{J})} = \begin{pmatrix} 12.788 \\ 6.26 \end{pmatrix}$$

$$\sqrt{\text{eigenvals}(\mathbf{J}^{-1} \cdot \mathbf{K})} = \begin{pmatrix} 12.788 \\ 6.26 \end{pmatrix}$$

$$\Delta(\omega) := \mathbf{K} - \omega^2 \cdot \mathbf{J}$$

$$\det(\omega) := |\Delta(\omega)|$$

$$\omega := 0, 0.1..15$$



$$\omega := \sqrt{\text{genvals}(\mathbf{K}, \mathbf{J})} = \begin{pmatrix} 12.788 \\ 6.26 \end{pmatrix}$$

$$\Phi := \text{genvecs}(\mathbf{K}, \mathbf{J}) = \begin{pmatrix} 1 & 0.218 \\ -0.07 & 1 \end{pmatrix}$$

$$\mathbf{X}_1 := \Phi^{\langle 1 \rangle} = \begin{pmatrix} 1 \\ -0.07 \end{pmatrix} \quad \omega_1 = 12.788$$

$$\mathbf{X}_2 := \Phi^{\langle 2 \rangle} = \begin{pmatrix} 0.218 \\ 1 \end{pmatrix} \quad \omega_2 = 6.26$$

$$\mathbf{J}_{\text{star}} := \Phi^T \cdot \mathbf{J} \cdot \Phi = \begin{pmatrix} 1.218 & 0 \\ 0 & 3.783 \end{pmatrix}$$

$$\mathbf{K}_{\text{star}} := \Phi^T \cdot \mathbf{K} \cdot \Phi = \begin{pmatrix} 199.24 & 0 \\ 0 & 148.262 \end{pmatrix}$$

$$\omega_1 = 12.788 \quad \sqrt{\frac{\mathbf{K}_{\text{star}}_{1,1}}{\mathbf{J}_{\text{star}}_{1,1}}} = 12.788$$

$$\omega_2 = 6.26 \quad \sqrt{\frac{\mathbf{K}_{\text{star}}_{2,2}}{\mathbf{J}_{\text{star}}_{2,2}}} = 6.26$$

$$\mu_1 := \frac{1}{\sqrt{\mathbf{X}_1^T \cdot \mathbf{J} \cdot \mathbf{X}_1}} = 0.906$$

$$\mathbf{X}_1 = \begin{pmatrix} 1 \\ -0.07 \end{pmatrix}$$

$$\mathbf{X}_1 := \mu_1 \cdot \mathbf{X}_1 = \begin{pmatrix} 0.906 \\ -0.064 \end{pmatrix}$$

$$\Phi = \begin{pmatrix} 1 & 0.218 \\ -0.07 & 1 \end{pmatrix}$$

$$\mu_2 := \frac{1}{\sqrt{\mathbf{X}_2^T \cdot \mathbf{J} \cdot \mathbf{X}_2}} = 0.514$$

$$\mathbf{X}_2 = \begin{pmatrix} 0.218 \\ 1 \end{pmatrix}$$

$$\mathbf{X}_2 := \mu_2 \cdot \mathbf{X}_2 = \begin{pmatrix} 0.112 \\ 0.514 \end{pmatrix}$$

$$\Phi := \text{augment}(\mathbf{X}_1, \mathbf{X}_2) = \begin{pmatrix} 0.906 & 0.112 \\ -0.064 & 0.514 \end{pmatrix}$$

$$\Phi^T \mathbf{J} \Phi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{W} := \Phi^T \cdot \mathbf{K} \cdot \Phi = \begin{pmatrix} 163.537 & 0 \\ 0 & 39.193 \end{pmatrix} \quad (\omega_1)^2 = 163.537 \quad (\omega_2)^2 = 39.193$$

Forzante (costante)

$$F := 500$$

$$\mathbf{F} := \begin{pmatrix} 0 \\ F \cdot c \end{pmatrix} \quad \mathbf{F} = \begin{pmatrix} 0 \\ 110 \end{pmatrix}$$

$$\Phi^T = \begin{pmatrix} 0.906 & -0.064 \\ 0.112 & 0.514 \end{pmatrix}$$

$$\mathbf{Q} := \Phi^T \cdot \mathbf{F} = \begin{pmatrix} -6.988 \\ 56.556 \end{pmatrix}$$

Equazioni di moto in formato matriciale (nelle coord. principali  $p_1$  e  $p_2$ )

$$\begin{pmatrix} p_1''(t) \\ p_2''(t) \end{pmatrix} + \begin{bmatrix} (\omega_1)^2 & 0 \\ 0 & (\omega_2)^2 \end{bmatrix} \cdot \begin{pmatrix} p_1(t) \\ p_2(t) \end{pmatrix} = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}$$

Equazioni differenziali scritte separatamente nelle coordinate principali  $p_1(t)$  e  $p_2(t)$

$$p_1''(t) + \omega_1^2 \cdot p_1(t) = Q_1$$

$$(\omega_1)^2 = 163.537$$

$$Q_1 = -6.988$$

$$p_2''(t) + \omega_2^2 \cdot p_2(t) = Q_2$$

$$(\omega_2)^2 = 39.193$$

$$Q_2 = 56.556$$

$$\mathbf{p}(t) = \begin{pmatrix} p_1(t) \\ p_2(t) \end{pmatrix}$$

Soluzione delle equazioni in coord. principali (dalla teoria dei sistemi a 1 gdl non smorzati):

$$p_1(t) = A_1 \cdot \cos(\omega_1 \cdot t) + B_1 \cdot \sin(\omega_1 \cdot t) + \frac{Q_1}{(\omega_1)^2}$$

$$p_2(t) = A_2 \cdot \cos(\omega_2 \cdot t) + B_2 \cdot \sin(\omega_2 \cdot t) + \frac{Q_2}{(\omega_2)^2}$$

Derivate temporali prime (velocità); sono state calcolate a mano, per semplicità

$$p'_1(t) = -\omega_1 \cdot A_1 \cdot \sin(\omega_1 \cdot t) + \omega_1 \cdot B_1 \cdot \cos(\omega_1 \cdot t)$$

$$p'_2(t) = -\omega_2 \cdot A_2 \cdot \sin(\omega_2 \cdot t) + \omega_2 \cdot B_2 \cdot \cos(\omega_2 \cdot t)$$

Occorre assegnare le condizioni iniziali per calcolare le costanti  $A_1, B_1, A_2, B_2$

Le condizioni iniziali vengono sempre assegnate nelle coordinate fisiche; occorre quindi convertirle nelle coordinate principali.

Si utilizzano le seguenti trasformazioni:

$$\mathbf{p}(t) = \mathbf{\Phi}^{-1} \cdot \mathbf{x}(t) \quad \text{per le posizioni}$$

$$\mathbf{p}'(t) = \mathbf{\Phi}^{-1} \cdot \mathbf{x}'(t) \quad \text{per le velocità}$$

All'istante iniziale  $t=0$  si avrà:

$$\mathbf{p}(0) = \mathbf{\Phi}^{-1} \cdot \mathbf{x}(0)$$

$$\mathbf{p}'(0) = \mathbf{\Phi}^{-1} \cdot \mathbf{x}'(0)$$

Assegniamo le condizioni iniziali nelle coordinate fisiche

$$\mathbf{x}(0) = \begin{pmatrix} \alpha_0 \\ \beta_0 \end{pmatrix}$$

$$\mathbf{x}'(0) = \begin{pmatrix} \alpha'_0 \\ \beta'_0 \end{pmatrix}$$

$$\alpha_0 := 20 \cdot \text{deg} = 0.349 \cdot \text{rad}$$

$$\beta_0 := 10 \cdot \text{deg} = 0.175 \cdot \text{rad}$$

$$\alpha'_0 := 2$$

$$\beta'_0 := -3$$

Convertiamo le condizioni iniziali nelle coordinate principali

Per le posizioni:

$$\mathbf{p}(0) = \mathbf{\Phi}^{-1} \cdot \mathbf{x}(0)$$

$$\begin{pmatrix} p_{10} \\ p_{20} \end{pmatrix} := \mathbf{\Phi}^{-1} \cdot \begin{pmatrix} \alpha_0 \\ \beta_0 \end{pmatrix} = \begin{pmatrix} 0.338 \\ 0.381 \end{pmatrix}$$

$$p_{10} = 0.338$$

$$p_{20} = 0.381$$

Per le velocità:

$$\mathbf{p}'(0) = \mathbf{\Phi}^{-1} \cdot \mathbf{x}'(0)$$

$$\begin{pmatrix} p'_{10} \\ p'_{20} \end{pmatrix} := \mathbf{\Phi}^{-1} \cdot \begin{pmatrix} \alpha'_0 \\ \beta'_0 \end{pmatrix} = \begin{pmatrix} 2.884 \\ -5.478 \end{pmatrix}$$

$$p'_{10} = 2.884$$

$$p'_{20} = -5.478$$

Imponendo le condizioni iniziali si ha:

$$p_1(0) = A_1 + \frac{Q_1}{(\omega_1)^2} = p_{10}$$

quindi

$$A_1 := p_{10} - \frac{Q_1}{(\omega_1)^2} = 0.381$$

$$p_2(0) = A_2 + \frac{Q_2}{(\omega_2)^2} = p_{20}$$

quindi

$$A_2 := p_{20} - \frac{Q_2}{(\omega_2)^2} = -1.062$$

$$p'_1(0) = \omega_1 \cdot B_1 = p'_{10}$$

quindi

$$B_1 := \frac{p'_{10}}{\omega_1} = 0.226$$

$$p'_2(0) = \omega_2 \cdot B_2 = p'_{20}$$

quindi

$$B_2 := \frac{p'_{20}}{\omega_2} = -0.875$$

La soluzione in coordinate principali è ora calcolabile, perché sono state determinate le costanti  $A_1, B_1, A_2, B_2$

$$p_1(t) := A_1 \cdot \cos(\omega_1 \cdot t) + B_1 \cdot \sin(\omega_1 \cdot t) + \frac{Q_1}{(\omega_1)^2}$$

$$p_2(t) := A_2 \cdot \cos(\omega_2 \cdot t) + B_2 \cdot \sin(\omega_2 \cdot t) + \frac{Q_2}{(\omega_2)^2}$$

Occorre infine applicare la formula 5.195 per calcolare la soluzione nelle coordinate fisiche  $\alpha$  e  $\beta$

$$\mathbf{p}(t) := \begin{pmatrix} p_1(t) \\ p_2(t) \end{pmatrix}$$

$$\mathbf{x}(t) := \mathbf{\Phi} \cdot \mathbf{p}(t)$$

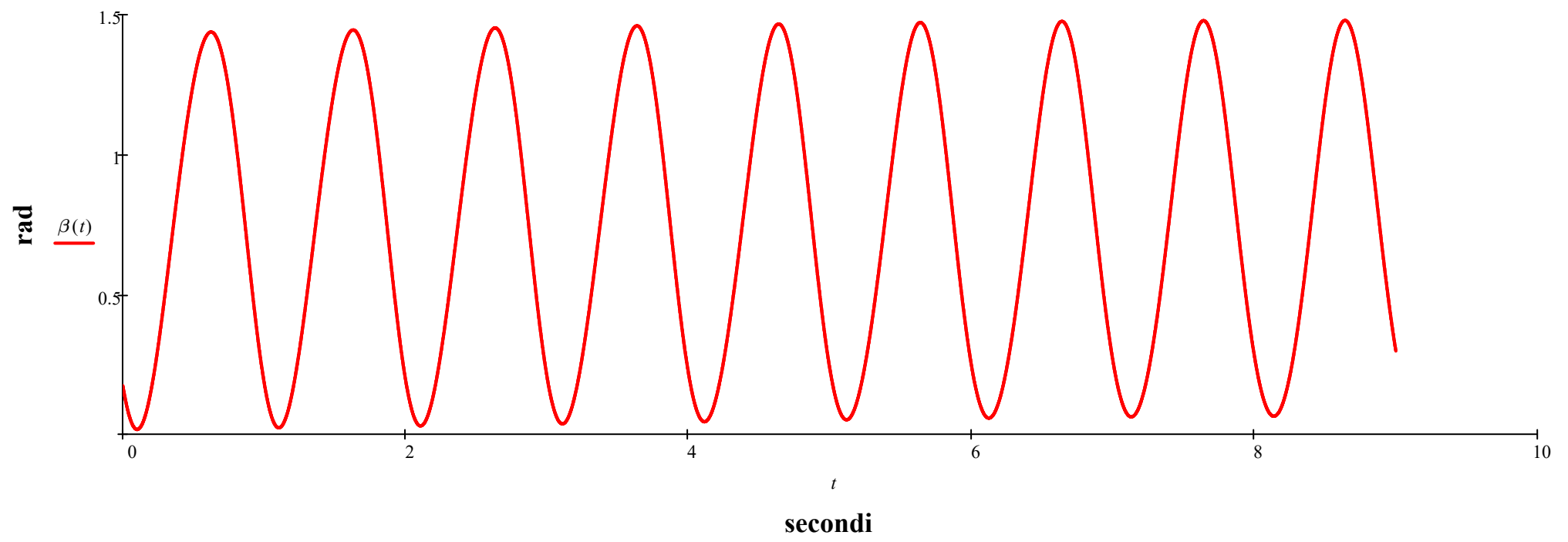
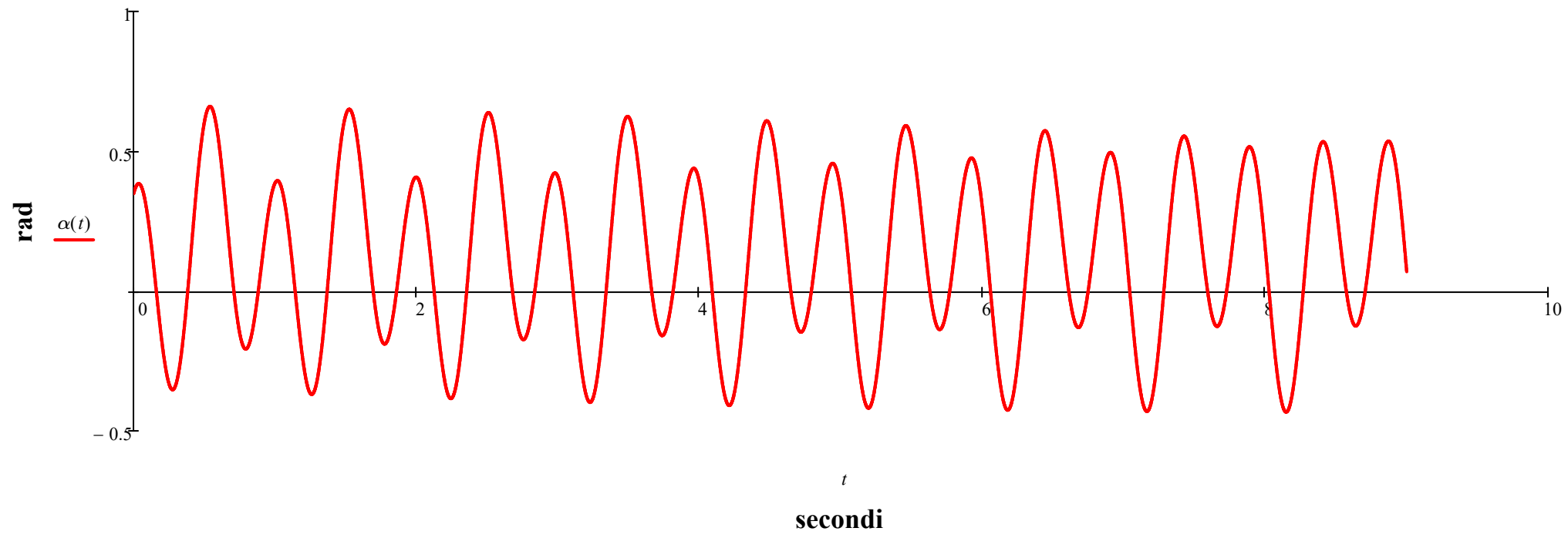
$$\alpha(t) := \mathbf{X}(t)_1$$

$$\beta(t) := \mathbf{X}(t)_2$$

$$T_{max} := 9$$

$$\Delta t := 10^{-3}$$

$$t := 0, \Delta t .. T_{max}$$



Verifica dei calcoli mediante integrazione numerica

$$\mathbf{I} := \text{identity}(2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{O} := \mathbf{0} \cdot \mathbf{I} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\mathbf{C} := \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Assenza di smorzamento



$$\mathbf{A}_{\text{sup}} := \text{augment}(\mathbf{O}, \mathbf{I}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{A}_{\text{inf}} := \text{augment}(-\mathbf{J}^{-1} \cdot \mathbf{K}, -\mathbf{J}^{-1} \cdot \mathbf{C}) = \begin{pmatrix} -161.667 & 26.667 & 0 & 0 \\ 8.588 & -41.063 & 0 & 0 \end{pmatrix}$$

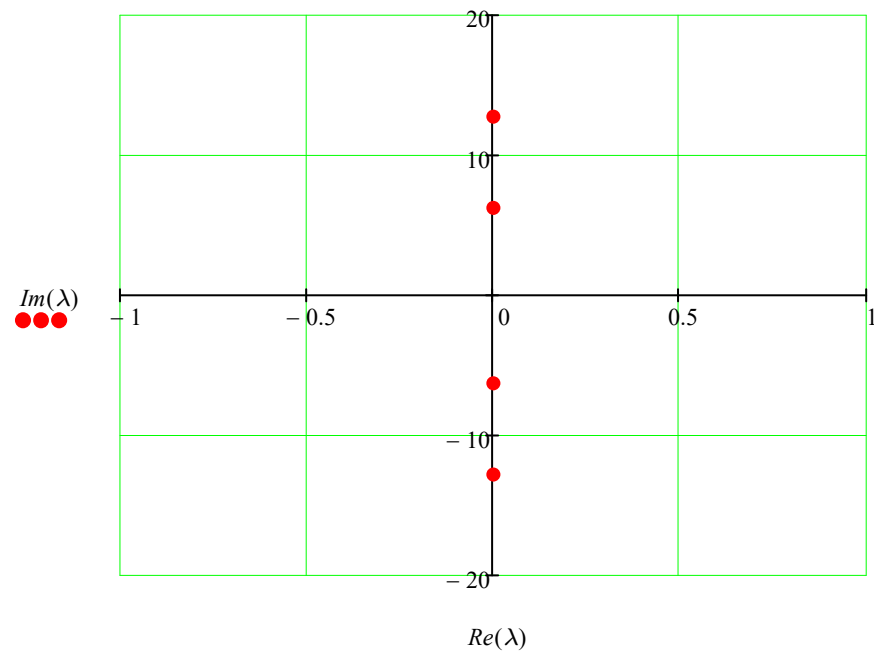
$$\mathbf{A} := \text{stack}(\mathbf{A}_{\text{sup}}, \mathbf{A}_{\text{inf}}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -161.667 & 26.667 & 0 & 0 \\ 8.588 & -41.063 & 0 & 0 \end{pmatrix}$$

$$\mathbf{o} := \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\mathbf{F} = \begin{pmatrix} 0 \\ 110 \end{pmatrix}$$

$$\mathbf{b} := \text{stack}(\mathbf{o}, \mathbf{J}^{-1} \cdot \mathbf{F}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 29.522 \end{pmatrix}$$

$$\lambda := \text{eigenvals}(\mathbf{A}) = \begin{pmatrix} 12.788i \\ -12.788i \\ 6.26i \\ -6.26i \end{pmatrix} \quad \omega = \begin{pmatrix} 12.788 \\ 6.26 \end{pmatrix}$$



$$\mathbf{y} := \begin{pmatrix} \alpha_0 \\ \beta_0 \\ \alpha'_0 \\ \beta'_0 \end{pmatrix} = \begin{pmatrix} 0.349 \\ 0.175 \\ 2 \\ -3 \end{pmatrix}$$

Condizioni iniziali

$$\Delta t = 1 \times 10^{-3}$$

$$N := \text{ceil}\left(\frac{T_{max}}{\Delta t}\right) = 9000$$

$$EQMOTO(t, \mathbf{y}) := \mathbf{A} \cdot \mathbf{y} + \mathbf{b}$$

$$TAB := \text{rkfixed}(\mathbf{y}, 0, T_{max}, N, EQMOTO)$$

$$tempo := TAB^{(1)}$$

$$alfa := TAB^{(2)}$$

$$beta := TAB^{(3)}$$

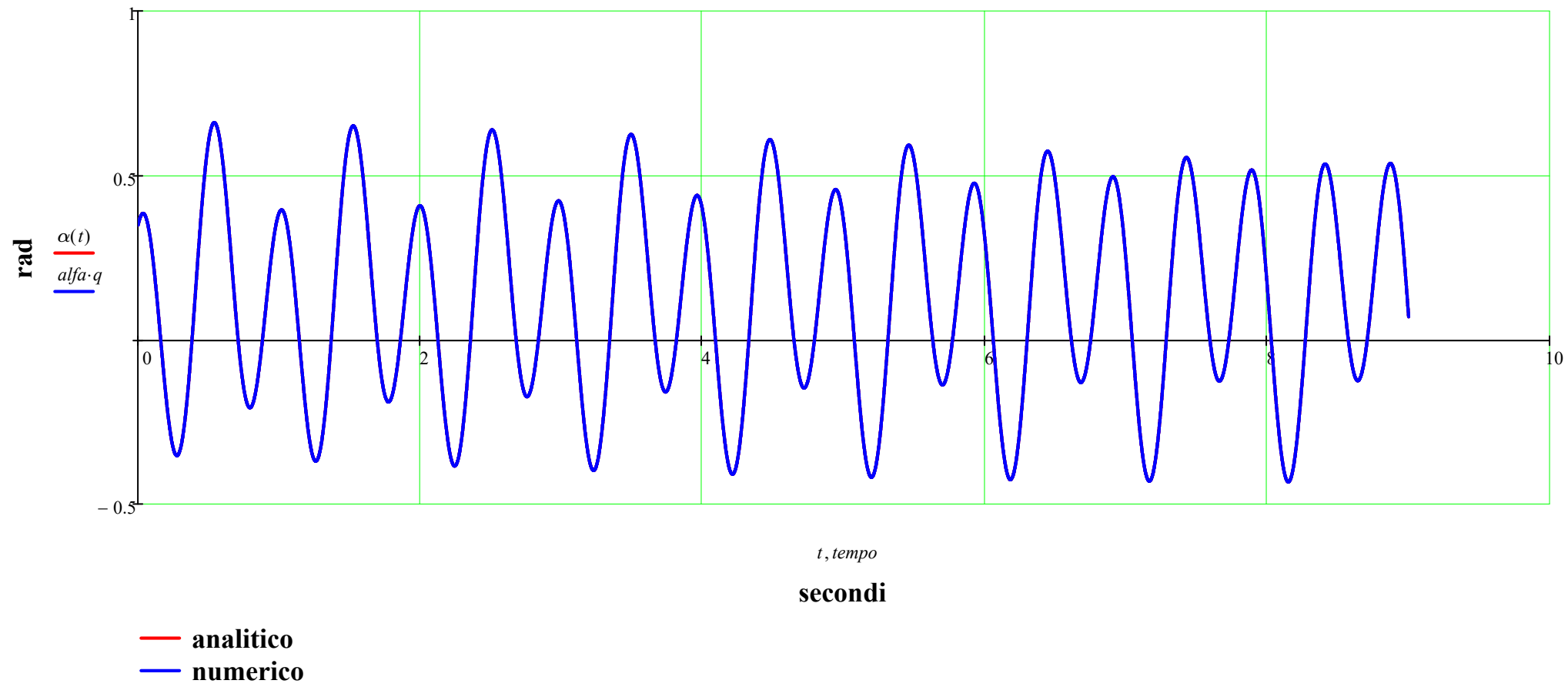
$$q := 1$$

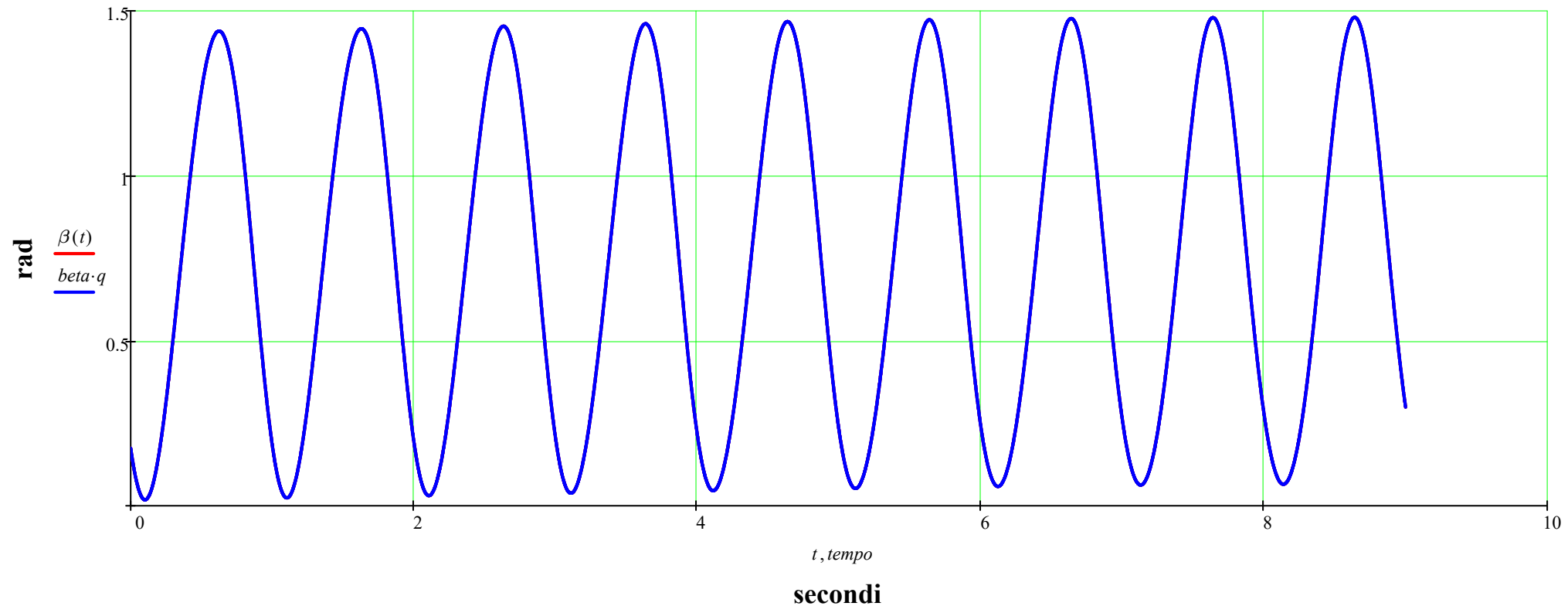
Equazioni di moto nello spazio di stato

Soluzione per via numerica (Runge-Kutta)

Se il procedimento è corretto le due tracce (metodo analitico e numerico) devono sempre sovrapporsi

Variabile on/off





— analitico  
— numerico