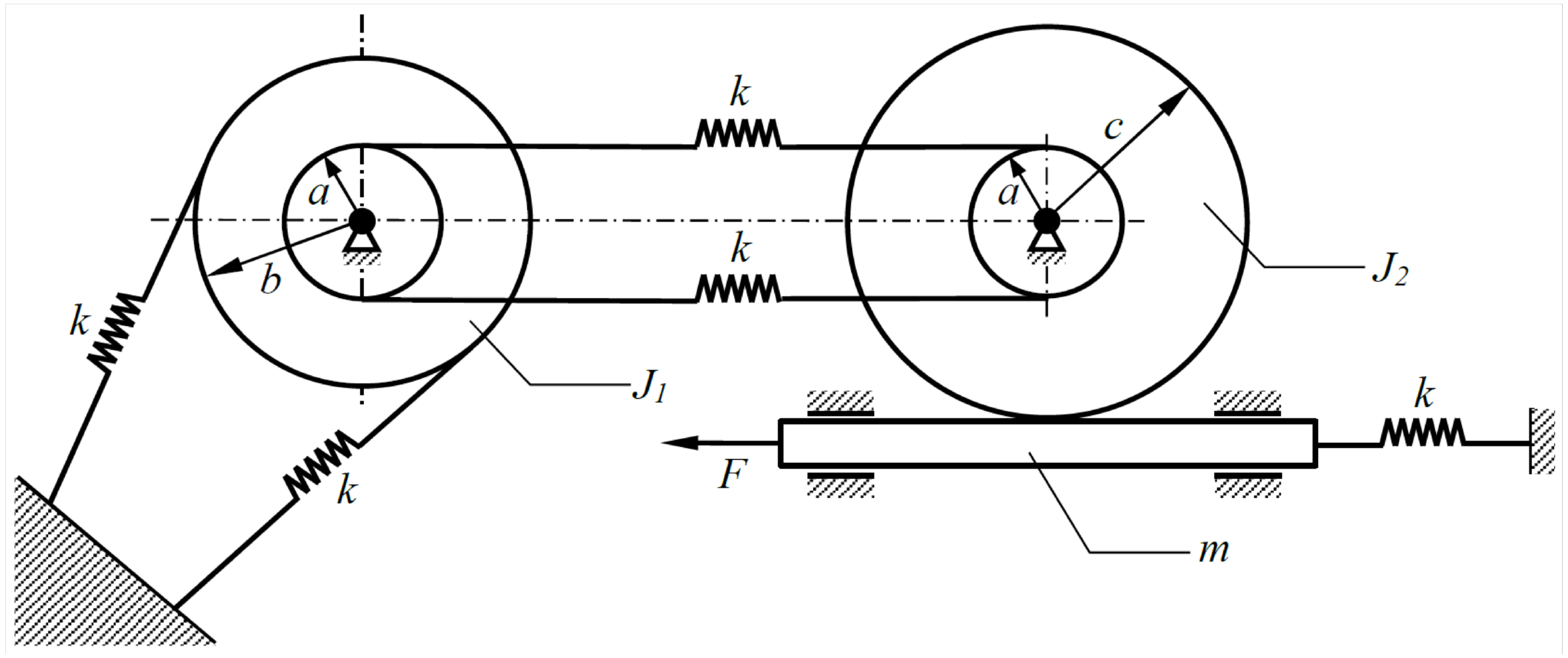
Uso delle coordinate principali: sistema a due gdl non smorzato



Parametri geometrici

$a := 80 \cdot 10^{-3} = 0.08$

$b := 180 \cdot 10^{-3} = 0.18$

$c := 220 \cdot 10^{-3} = 0.22$

Parametri inerziali

$J_1 := 1.2$

$J_2 := 3$

$m := 15$

Parametri elastici

$k := 2500$

Energia potenziale

$$V(\alpha,\beta) := \frac{1}{2}\cdot\left[2\cdot k\cdot b^2\cdot\alpha^2 + 2\cdot k\cdot(a\cdot\alpha - a\cdot\beta)^2 + k\cdot c^2\cdot\beta^2\right]$$

$$\frac{\partial}{\partial\alpha}V(\alpha,\beta)\ simplify\ \rightarrow 194.0\cdot\alpha - 32.0\cdot\beta$$

$$\frac{\partial}{\partial\beta}V(\alpha,\beta)\ simplify\ \rightarrow 153.0\cdot\beta - 32.0\cdot\alpha$$

Matrici di massa e di rigidezza

$$\mathbf{J} := \begin{bmatrix} J_1 & 0 \\ 0 & \left(J_2 + m\cdot c^2\right) \end{bmatrix} = \begin{pmatrix} 1.2 & 0 \\ 0 & 3.726 \end{pmatrix}$$

$$diag\left[\begin{bmatrix} J_1 \\ \left(J_2 + m\cdot c^2\right) \end{bmatrix}\right] = \begin{pmatrix} 1.2 & 0 \\ 0 & 3.726 \end{pmatrix}$$

$$\mathbf{K} := \begin{bmatrix} 2\cdot k\cdot\left(a^2 + b^2\right) & -2\cdot k\cdot a^2 \\ -2\cdot k\cdot a^2 & k\cdot\left(2\cdot a^2 + c^2\right) \end{bmatrix} = \begin{pmatrix} 194 & -32 \\ -32 & 153 \end{pmatrix}$$

Calcolo delle pulsazioni proprie

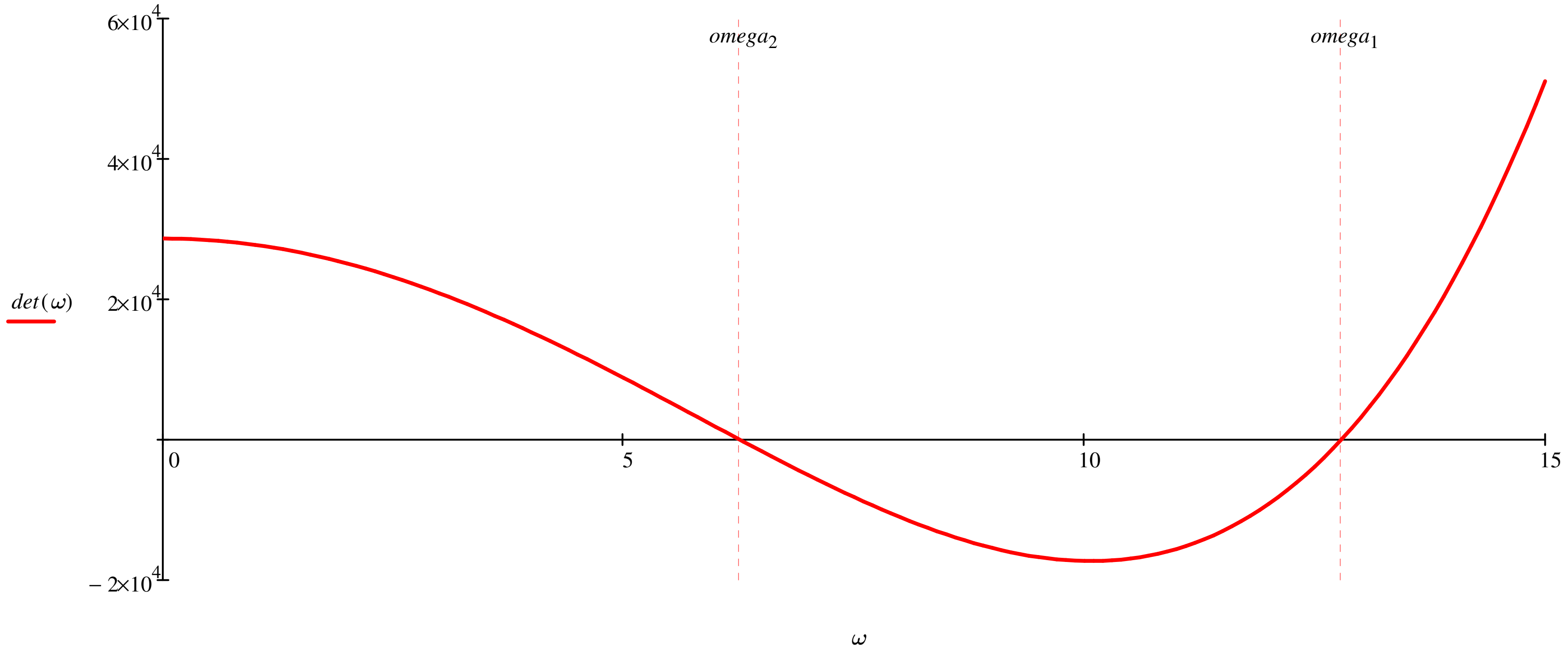
$$omega := \sqrt{genvals(\mathbf{K},\mathbf{J})} = \begin{pmatrix} 12.788 \\ 6.26 \end{pmatrix}$$

$$\sqrt{eigenvals\left(\mathbf{J}^{-1}\cdot\mathbf{K}\right)} = \begin{pmatrix} 12.788 \\ 6.26 \end{pmatrix}$$

$$\Delta(\omega) := \mathbf{K} - \omega^2\cdot\mathbf{J}$$

$$det(\omega) := \left|\Delta(\omega)\right|$$

$$\omega := 0, 0.1 .. 15$$



$\omega := \sqrt{genvals(\mathbf{K},\mathbf{J})} = \begin{pmatrix} 12.788 \\ 6.26 \end{pmatrix}$

$\mathbf{\Phi} := genvecs(\mathbf{K},\mathbf{J}) = \begin{pmatrix} 1 & 0.218 \\ -0.07 & 1 \end{pmatrix}$

$\mathbf{X_1} := \mathbf{\Phi}^{\langle 1 \rangle} = \begin{pmatrix} 1 \\ -0.07 \end{pmatrix}$ $\omega_1 = 12.788$

$\mathbf{X_2} := \mathbf{\Phi}^{\langle 2 \rangle} = \begin{pmatrix} 0.218 \\ 1 \end{pmatrix}$ $\omega_2 = 6.26$

$\mathbf{J_{star}} := \mathbf{\Phi}^T \cdot \mathbf{J} \cdot \mathbf{\Phi} = \begin{pmatrix} 1.218 & 0 \\ 0 & 3.783 \end{pmatrix}$

$\mathbf{K_{star}} := \mathbf{\Phi}^T \cdot \mathbf{K} \cdot \mathbf{\Phi} = \begin{pmatrix} 199.24 & 0 \\ 0 & 148.262 \end{pmatrix}$

$\omega_1 = 12.788$ $\sqrt{\dfrac{\mathbf{K_{star}}_{1,1}}{\mathbf{J_{star}}_{1,1}}} = 12.788$

$\omega_2 = 6.26$ $\sqrt{\dfrac{\mathbf{K_{star}}_{2,2}}{\mathbf{J_{star}}_{2,2}}} = 6.26$

$\mu_1 := \dfrac{1}{\sqrt{\mathbf{X_1}^T \cdot \mathbf{J} \cdot \mathbf{X_1}}} = 0.906$ $\mu_2 := \dfrac{1}{\sqrt{\mathbf{X_2}^T \cdot \mathbf{J} \cdot \mathbf{X_2}}} = 0.514$

$\mathbf{X_1} = \begin{pmatrix} 1 \\ -0.07 \end{pmatrix}$ $\mathbf{X_2} = \begin{pmatrix} 0.218 \\ 1 \end{pmatrix}$

$\mathbf{X_1} := \mu_1 \cdot \mathbf{X_1} = \begin{pmatrix} 0.906 \\ -0.064 \end{pmatrix}$ $\mathbf{X_2} := \mu_2 \cdot \mathbf{X_2} = \begin{pmatrix} 0.112 \\ 0.514 \end{pmatrix}$

$\mathbf{\Phi} = \begin{pmatrix} 1 & 0.218 \\ -0.07 & 1 \end{pmatrix}$

$$\mathbf{\Phi} := augment(\mathbf{X_1}, \mathbf{X_2}) = \begin{pmatrix} 0.906 & 0.112 \\ -0.064 & 0.514 \end{pmatrix}$$

$$\mathbf{\Phi}^T \cdot \mathbf{J} \cdot \mathbf{\Phi} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{W} := \mathbf{\Phi}^T \cdot \mathbf{K} \cdot \mathbf{\Phi} = \begin{pmatrix} 163.537 & 0 \\ 0 & 39.193 \end{pmatrix} \qquad (\omega_1)^2 = 163.537 \qquad (\omega_2)^2 = 39.193$$

Forzante (costante)

$F := 500$

$$\mathbf{F} := \begin{pmatrix} 0 \\ F \cdot c \end{pmatrix} \qquad \mathbf{F} = \begin{pmatrix} 0 \\ 110 \end{pmatrix}$$

$$\mathbf{\Phi}^T = \begin{pmatrix} 0.906 & -0.064 \\ 0.112 & 0.514 \end{pmatrix}$$

$$\mathbf{Q} := \mathbf{\Phi}^T \cdot \mathbf{F} = \begin{pmatrix} -6.988 \\ 56.556 \end{pmatrix}$$

Equazioni di moto in formato matriciale (nelle coord. principali p_1 e p_2)

$$\begin{pmatrix} p''_1(t) \\ p''_2(t) \end{pmatrix} + \begin{bmatrix} (\omega_1)^2 & 0 \\ 0 & (\omega_2)^2 \end{bmatrix} \cdot \begin{pmatrix} p_1(t) \\ p_2(t) \end{pmatrix} = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}$$

Equazioni differenziali scritte separatamente nelle coordinate principali p_1(t) e p_2(t)

$$p''_1(t) + \omega_1{}^2 \cdot p_1(t) = Q_1 \qquad (\omega_1)^2 = 163.537 \qquad \mathbf{Q}_1 = -6.988$$

$$p''_2(t) + \omega_2{}^2 \cdot p_2(t) = Q_2 \qquad (\omega_2)^2 = 39.193 \qquad \mathbf{Q}_2 = 56.556$$

$$\mathbf{p}(t) = \begin{pmatrix} p_1(t) \\ p_2(t) \end{pmatrix}$$

Soluzione delle equazioni in coord. principali (dalla teoria dei sistemi a 1 gdl non smorzati):

$$p_1(t) = A_1 \cdot cos(\omega_1 \cdot t) + B_1 \cdot sin(\omega_1 \cdot t) + \frac{\mathbf{Q}_1}{(\omega_1)^2}$$

$$p_2(t) = A_2 \cdot cos(\omega_2 \cdot t) + B_2 \cdot sin(\omega_2 \cdot t) + \frac{\mathbf{Q}_2}{(\omega_2)^2}$$

Derivate temporali prime (velocità); sono state calcolate a mano, per semplicità

$$p'_1(t) = -\omega_1 \cdot A_1 \cdot sin(\omega_1 \cdot t) + \omega_1 \cdot B_1 \cdot cos(\omega_1 \cdot t)$$

$$p'_2(t) = -\omega_2 \cdot A_2 \cdot sin(\omega_2 \cdot t) + \omega_2 \cdot B_2 \cdot cos(\omega_2 \cdot t)$$

Occorre assegnare le condizioni iniziali per calcolare le costanti A_1, B_1, A_2, B_2

Le condizioni iniziali vengono sempre assegnate nelle coordinate fisiche; occorre quindi convertirle nelle coordinate principali.

Si utilizzano le seguenti trasformazioni:

$\mathbf{p}(t) = \mathbf{\Phi}^{-1} \cdot \mathbf{x}(t)$ per le posizioni

$\mathbf{p'}(t) = \mathbf{\Phi}^{-1} \cdot \mathbf{x'}(t)$ per le velocità

All'istante iniziale t=0 si avrà:

$$\mathbf{p}(0) = \mathbf{\Phi}^{-1} \cdot \mathbf{x}(0)$$

$$\mathbf{p'}(0) = \mathbf{\Phi}^{-1} \cdot \mathbf{x'}(0)$$

Assegniamo le condizioni iniziali nelle coordinate fisiche

$$\mathbf{x}(0) = \begin{pmatrix} \alpha_0 \\ \beta_0 \end{pmatrix}$$

$$\mathbf{x'}(0) = \begin{pmatrix} \alpha'_0 \\ \beta'_0 \end{pmatrix}$$

$\alpha_0 := 20 \cdot deg = 0.349 \cdot rad$

$\beta_0 := 10 \cdot deg = 0.175 \cdot rad$

$\alpha'_0 := 2$

$\beta'_0 := -3$

Convertiamo le condizioni iniziali nelle coordinate principali

Per le posizioni:

$$\mathbf{p}(0) = \mathbf{\Phi}^{-1} \cdot \mathbf{x}(0)$$

$$\begin{pmatrix} p_{10} \\ p_{20} \end{pmatrix} := \mathbf{\Phi}^{-1} \cdot \begin{pmatrix} \alpha_0 \\ \beta_0 \end{pmatrix} = \begin{pmatrix} 0.338 \\ 0.381 \end{pmatrix}$$

$p_{10} = 0.338$

$p_{20} = 0.381$

Per le velocità:

$$\mathbf{p'}(0) = \mathbf{\Phi}^{-1} \cdot \mathbf{x'}(0)$$

$$\begin{pmatrix} p'_{10} \\ p'_{20} \end{pmatrix} := \mathbf{\Phi}^{-1} \cdot \begin{pmatrix} \alpha'_0 \\ \beta'_0 \end{pmatrix} = \begin{pmatrix} 2.884 \\ -5.478 \end{pmatrix}$$

$p'_{10} = 2.884$

$p'_{20} = -5.478$

Imponendo le condizioni iniziali si ha:

$$p_1(0) = A_1 + \frac{\mathbf{Q}_1}{(\omega_1)^2} = p_{10}$$

quindi

$$A_1 := p_{10} - \frac{\mathbf{Q}_1}{(\omega_1)^2} = 0.381$$

$p_2(0) = A_2 + \frac{\mathbf{Q}_2}{(\omega_2)^2} = p_{20}$ quindi $A_2 := p_{20} - \frac{\mathbf{Q}_2}{(\omega_2)^2} = -1.062$

$p'_1(0) = \omega_1 \cdot B_1 = p'_{10}$ quindi $B_1 := \frac{p'_{10}}{\omega_1} = 0.226$

$p'_2(0) = \omega_2 \cdot B_2 = p'_{20}$ quindi $B_2 := \frac{p'_{20}}{\omega_2} = -0.875$

La soluzione in coordinate principali è ora calcolabile, perché sono state determinate le costanti A_1, B_1, A_2, B_2

$$p_1(t) := A_1 \cdot cos(\omega_1 \cdot t) + B_1 \cdot sin(\omega_1 \cdot t) + \frac{\mathbf{Q}_1}{(\omega_1)^2}$$

$$p_2(t) := A_2 \cdot cos(\omega_2 \cdot t) + B_2 \cdot sin(\omega_2 \cdot t) + \frac{\mathbf{Q}_2}{(\omega_2)^2}$$

Occorre infine applicare la formula 5.195 per calcolare la soluzione nelle coordinate fisiche α e β

$$\mathbf{p}(t) := \begin{pmatrix} p_1(t) \\ p_2(t) \end{pmatrix}$$

$$\mathbf{x}(t) := \Phi \cdot \mathbf{p}(t)$$

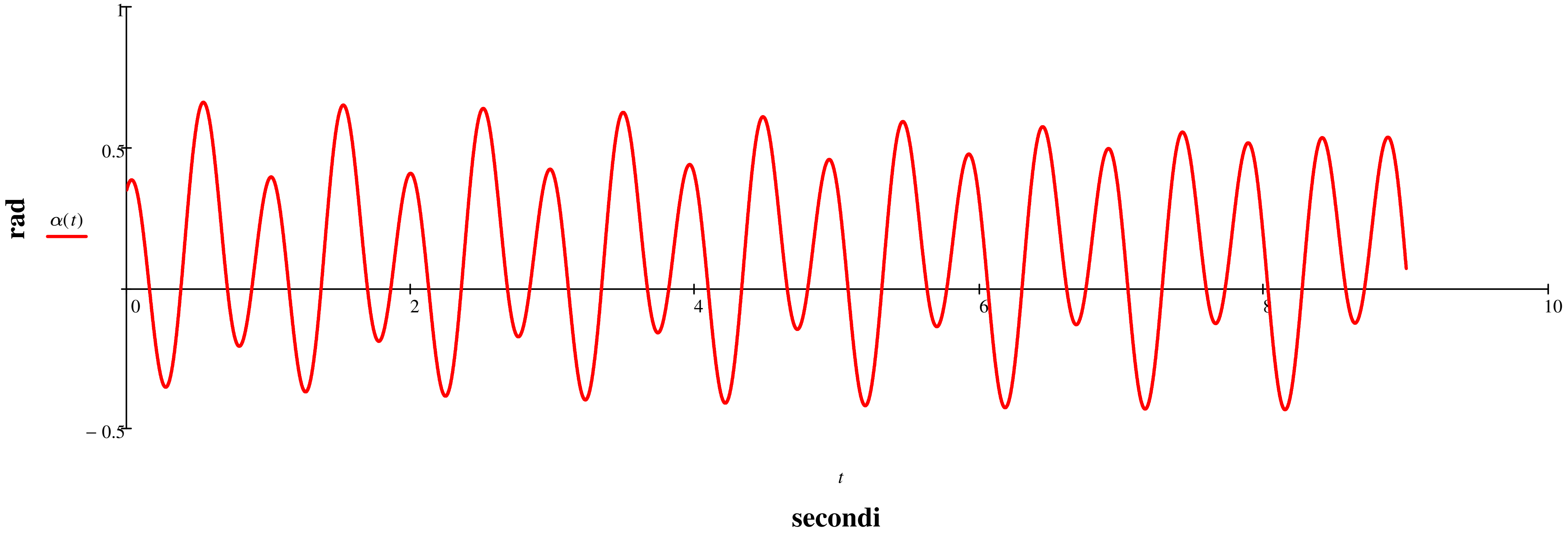
$$\alpha(t) := \mathbf{x}(t)_1$$

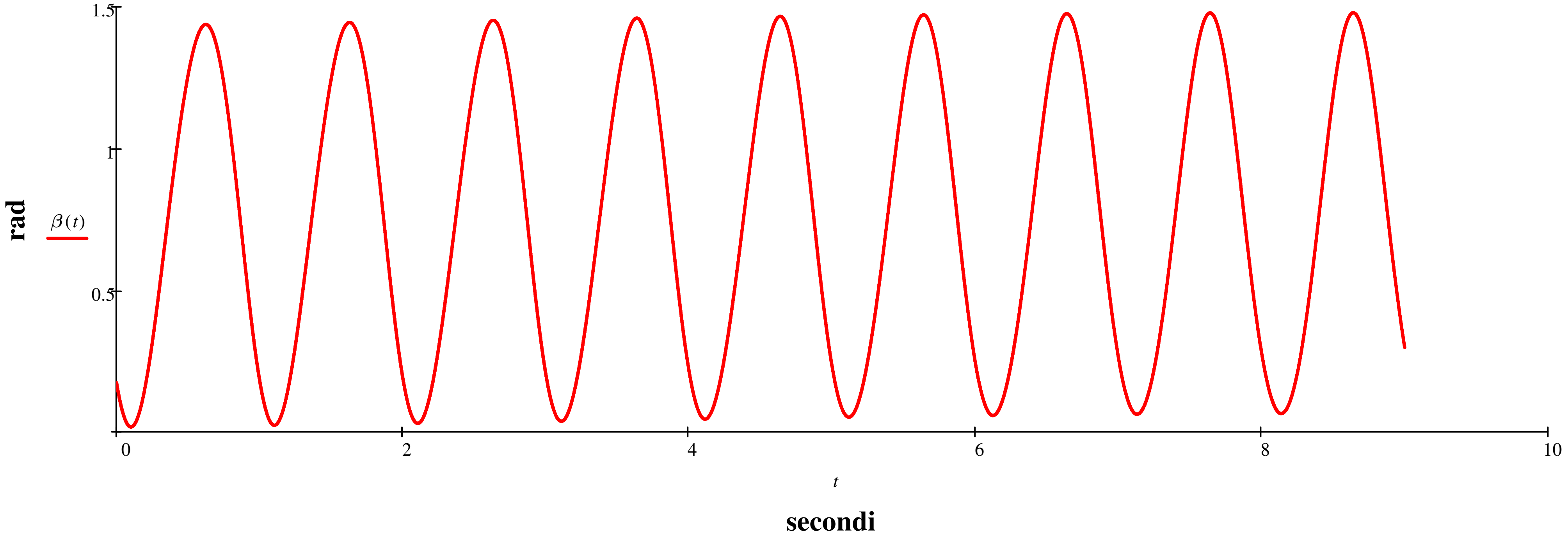
$$\beta(t) := \mathbf{x}(t)_2$$

$$T_{max} := 9$$

$$\Delta t := 10^{-3}$$

$$t := 0, \Delta t .. T_{max}$$





Verifica dei calcoli mediante integrazione numerica

$\mathbf{I} := identity(2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\mathbf{O} := 0 \cdot \mathbf{I} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\mathbf{C} := \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Assenza di smorzamento

$$\mathbf{A_{sup}} := augment(\mathbf{O},\mathbf{I}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{A_{inf}} := augment\left(-\mathbf{J}^{-1}\cdot\mathbf{K}, -\mathbf{J}^{-1}\cdot\mathbf{C}\right) = \begin{pmatrix} -161.667 & 26.667 & 0 & 0 \\ 8.588 & -41.063 & 0 & 0 \end{pmatrix}$$

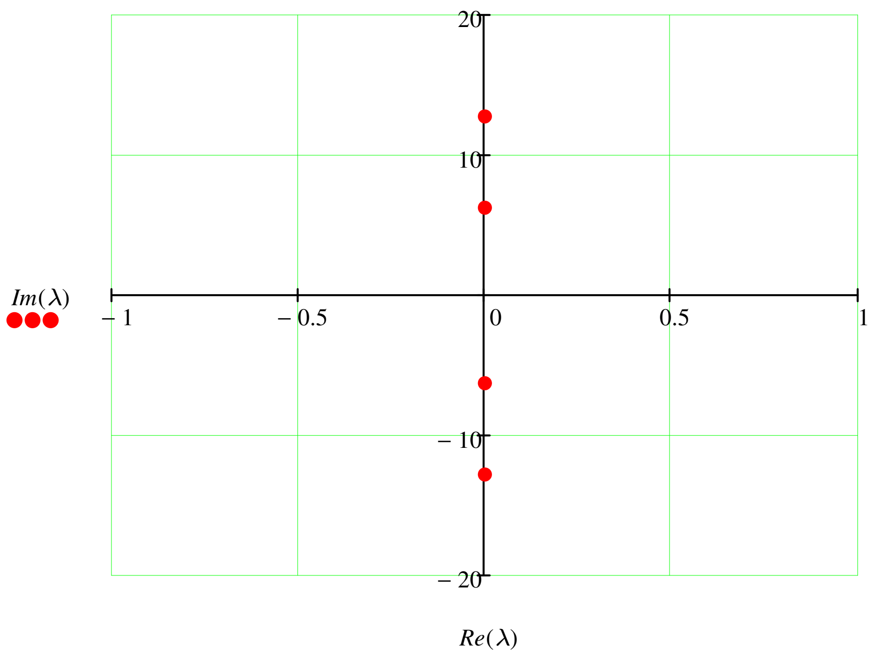
$$\mathbf{A} := stack\left(\mathbf{A_{sup}},\mathbf{A_{inf}}\right) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -161.667 & 26.667 & 0 & 0 \\ 8.588 & -41.063 & 0 & 0 \end{pmatrix}$$

$$\mathbf{o} := \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\mathbf{F} = \begin{pmatrix} 0 \\ 110 \end{pmatrix}$$

$$\mathbf{b} := stack\left(\mathbf{o},\mathbf{J}^{-1}\cdot\mathbf{F}\right) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 29.522 \end{pmatrix}$$

$$\lambda := eigenvals(\mathbf{A}) = \begin{pmatrix} 12.788i \\ -12.788i \\ 6.26i \\ -6.26i \end{pmatrix} \qquad \omega = \begin{pmatrix} 12.788 \\ 6.26 \end{pmatrix}$$



$$\mathbf{y} := \begin{pmatrix} \alpha_0 \\ \beta_0 \\ \alpha'_0 \\ \beta'_0 \end{pmatrix} = \begin{pmatrix} 0.349 \\ 0.175 \\ 2 \\ -3 \end{pmatrix}$$

Condizioni iniziali

$\Delta t = 1 \times 10^{-3}$

$N := \mathrm{ceil}\left(\frac{T_{max}}{\Delta t}\right) = 9000$

$EQMOTO(t, \mathbf{y}) := \mathbf{A} \cdot \mathbf{y} + \mathbf{b}$

Equazioni di moto nello spazio di stato

$TAB := rkfixed(\mathbf{y}, 0, T_{max}, N, EQMOTO)$

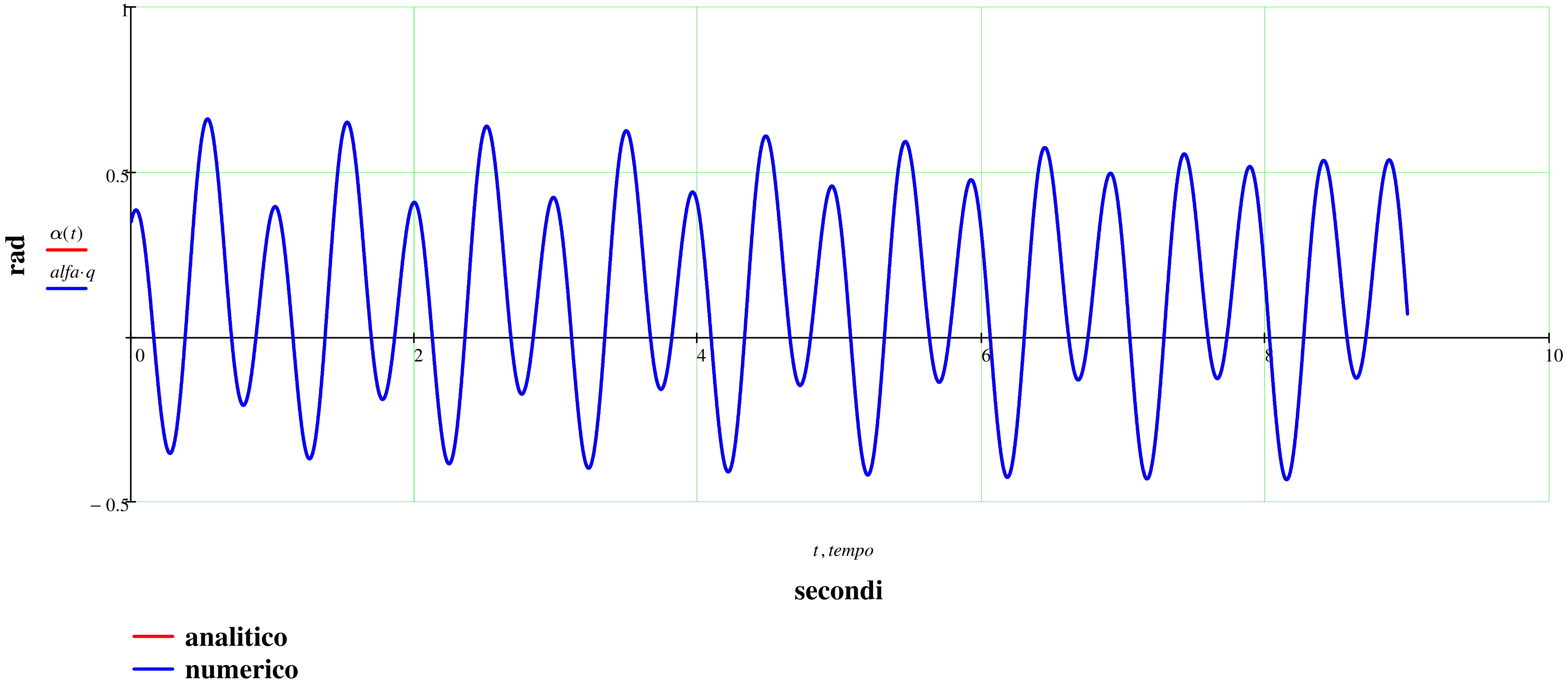
Soluzione per via numerica (Runge-Kutta)

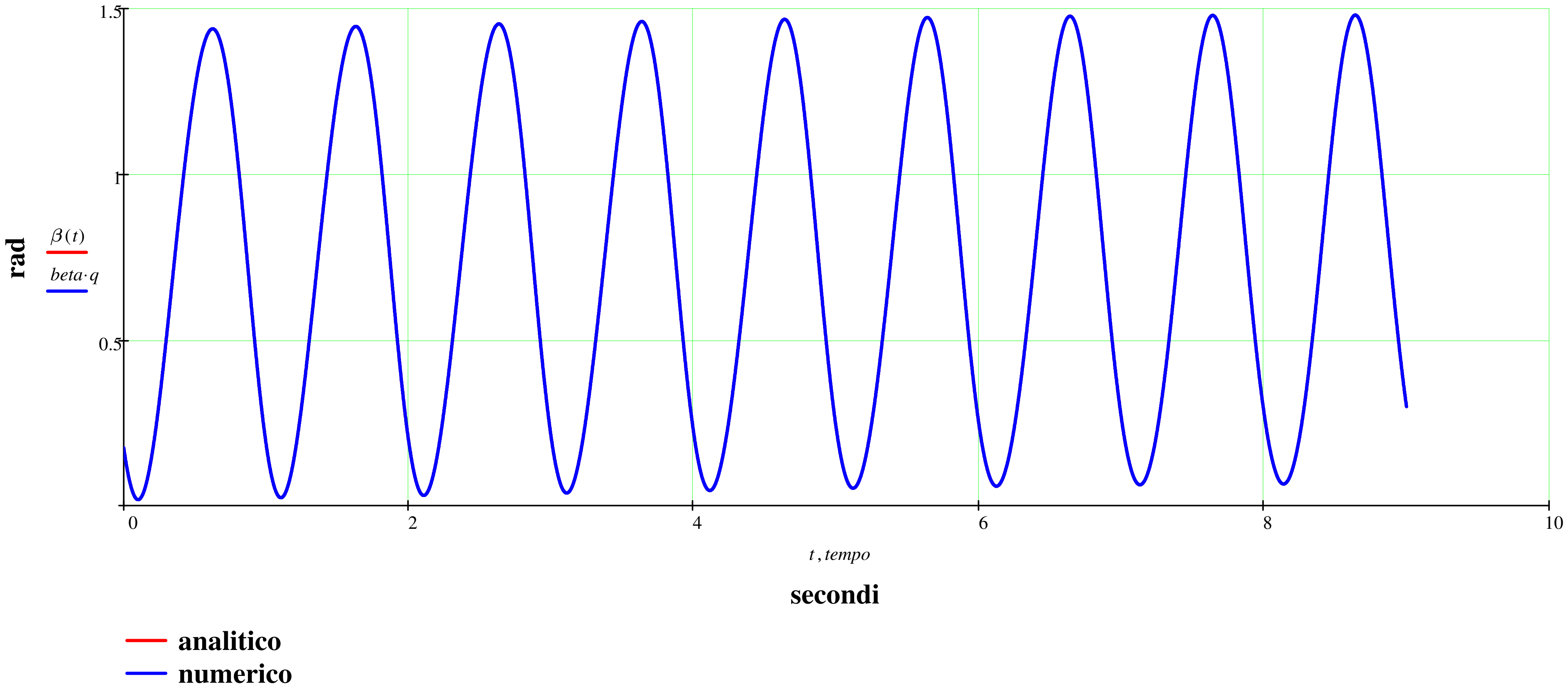
$tempo := TAB^{\langle 1 \rangle}$

Se il procedimento è corretto le due tracce (metodo analitico e numerico) devono sempre sovrapporsi

$alfa := TAB^{\langle 2 \rangle}$ $beta := TAB^{\langle 3 \rangle}$ $q := 1$

Variabile on/off




rad
β(t)
beta·q
1.5
1
0.5
0
2
4
6
8
10
t , tempo
secondi
analitico
numerico