Sistema a 3 gdl con smorzamento e con forzanti armoniche isofrequenziali

Parametri del sistema

$m_1 := 3$ $\quad c_1 := 5$ $\quad k_1 := 1000$

$m_2 := 5$ $\quad c_2 := 10$ $\quad k_2 := 1500$

$m_3 := 2$ $\quad c_3 := 5$ $\quad k_3 := 1400$

$c_4 := 12$ $\quad k_4 := 1800$

Condizioni iniziali

$x_{10} := 0.05$ $\quad x'_{10} := 1$

$x_{20} := 0.02$ $\quad x'_{20} := 2$

$x_{30} := 0.03$ $\quad x'_{30} := 0.5$

Durata simulazione e intervallo di campionamento

$T_{max} := 15$ $\quad \Delta t := 2 \cdot 10^{-3}$

Matrici

$$\mathbf{M} := diag\left(\begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}\right) \qquad \mathbf{M} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\mathbf{C} := \begin{bmatrix} (c_1 + c_2) & -c_2 & 0 \\ -c_2 & (c_2 + c_3) & -c_3 \\ 0 & -c_3 & (c_3 + c_4) \end{bmatrix} \qquad \mathbf{C} = \begin{pmatrix} 15 & -10 & 0 \\ -10 & 15 & -5 \\ 0 & -5 & 17 \end{pmatrix}$$

$$\mathbf{K} := \begin{bmatrix} (k_1 + k_2) & -k_2 & 0 \\ -k_2 & (k_2 + k_3) & -k_3 \\ 0 & -k_3 & (k_3 + k_4) \end{bmatrix} \qquad \mathbf{K} = \begin{pmatrix} 2500 & -1500 & 0 \\ -1500 & 2900 & -1400 \\ 0 & -1400 & 3200 \end{pmatrix}$$

Caratteristiche delle forzanti

$F_1 := 80$

$\psi_1 := 20 \cdot deg = 0.349 \cdot rad$

$F_2 := 60$

$\psi_2 := 10 \cdot deg = 0.175 \cdot rad$

$F_3 := 100$

$\psi_3 := -25 \cdot deg = -0.436 \cdot rad$

Pulsazione, frequenza e periodo delle forzanti

$\Omega := 5$ $\quad f := \dfrac{\Omega}{2 \cdot \pi} = 0.796$ $\quad T := \dfrac{1}{f} = 1.257$

$f_1(t) := F_1 \cdot sin(\Omega \cdot t + \psi_1)$

$f_2(t) := F_2 \cdot sin(\Omega \cdot t + \psi_2)$

$f_3(t) := F_3 \cdot sin(\Omega \cdot t + \psi_3)$

$$\mathbf{F}(t) := \begin{pmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \end{pmatrix} \qquad \mathbf{F}(t) \rightarrow \begin{pmatrix} 80 \cdot sin(5 \cdot t + 0.3490658503988659) \\ 60 \cdot sin(5 \cdot t + 0.17453292519943295) \\ 100 \cdot sin(5 \cdot t - 0.43633231299858238) \end{pmatrix}$$

Approccio numerico per la risoluzione delle equazioni differenziali di moto

$$\mathbf{y} := \begin{pmatrix} x_{10} \\ x_{20} \\ x_{30} \\ x'_{10} \\ x'_{20} \\ x'_{30} \end{pmatrix} = \begin{pmatrix} 0.05 \\ 0.02 \\ 0.03 \\ 1 \\ 2 \\ 0.5 \end{pmatrix}$$

Condizioni iniziali del sistema

$$N := ceil\left(\frac{T_{max}}{\Delta t}\right) = 7500 \qquad \Delta t = 2 \times 10^{-3}$$

$$\mathbf{M} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad \mathbf{K} = \begin{pmatrix} 2500 & -1500 & 0 \\ -1500 & 2900 & -1400 \\ 0 & -1400 & 3200 \end{pmatrix} \qquad \mathbf{C} = \begin{pmatrix} 15 & -10 & 0 \\ -10 & 15 & -5 \\ 0 & -5 & 17 \end{pmatrix}$$

$$\mathbf{I} := identity(3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{O} := 0 \cdot \mathbf{I} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{A_{sup}} := augment(\mathbf{O}, \mathbf{I}) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{A_{inf}} := augment\left(-\mathbf{M}^{-1} \cdot \mathbf{K}, -\mathbf{M}^{-1} \cdot \mathbf{C}\right) = \begin{pmatrix} -833.333 & 500 & 0 & -5 & 3.333 & 0 \\ 300 & -580 & 280 & 2 & -3 & 1 \\ 0 & 700 & -1600 & 0 & 2.5 & -8.5 \end{pmatrix}$$

$$\mathbf{A} := stack(\mathbf{A_{sup}}, \mathbf{A_{inf}}) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -833.333 & 500 & 0 & -5 & 3.333 & 0 \\ 300 & -580 & 280 & 2 & -3 & 1 \\ 0 & 700 & -1600 & 0 & 2.5 & -8.5 \end{pmatrix}$$

$$\mathbf{o} := \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \mathbf{F}(t) \rightarrow \begin{pmatrix} 80 \cdot sin(5 \cdot t + 0.3490658503988659) \\ 60 \cdot sin(5 \cdot t + 0.17453292519943295) \\ 100 \cdot sin(5 \cdot t - 0.43633231299858238) \end{pmatrix}$$

$$\mathbf{b}(t) := stack\left(\mathbf{o}, \mathbf{M}^{-1} \cdot \mathbf{F}(t)\right)$$

$$\mathbf{b}(t) \rightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{80 \cdot sin(5 \cdot t + 0.3490658503988659)}{3} \\ 12 \cdot sin(5 \cdot t + 0.17453292519943295) \\ 50 \cdot sin(5 \cdot t - 0.43633231299858238) \end{pmatrix}$$

$$EQMOTO(t, \mathbf{y}) := \mathbf{A} \cdot \mathbf{y} + \mathbf{b}(t)$$

Equazioni di moto nello spazio di stato

$$TAB := rkfixed(\mathbf{y}, 0, T_{max}, N, EQMOTO)$$

Soluzione in forma tabellare ottenuta con il metodo di Runge-Kutta

	1	2	3	4	5	6	7	8	9
1	0	0.05	0.02	0.03	1	2	0.5		
2	$2\cdot10^{-3}$	0.052	0.024	0.031	0.959	2.019	0.394		
3	$4\cdot10^{-3}$	0.054	0.028	0.032	0.92	2.035	0.294		
4	$6\cdot10^{-3}$	0.056	0.032	0.032	0.883	2.047	0.2		
5	$8\cdot10^{-3}$	0.057	0.036	0.032	0.848	2.056	0.113		
6	0.01	0.059	0.04	0.033	0.815	2.061	0.034		
7	0.012	0.061	0.045	0.033	0.785	2.062	-0.038		
8	0.014	0.062	0.049	0.032	0.756	2.06	-0.101		
9	0.016	0.064	0.053	0.032	0.73	2.053	-0.156		
10	0.018	0.065	0.057	0.032	0.707	2.042	-0.203		
11	0.02	0.066	0.061	0.031	0.686	2.028	-0.241		
12	0.022	0.068	0.065	0.031	0.667	2.009	-0.27		
13	0.024	0.069	0.069	0.03	0.65	1.986	-0.291		
14	0.026	0.07	0.073	0.03	0.636	1.96	-0.303		
15	0.028	0.072	0.077	0.029	0.624	1.93	-0.307		
16	0.03	0.073	0.081	0.028	0.614	1.896	...		

$TAB =$

$tempo := TAB^{\langle 1\rangle}$

$x_1 := TAB^{\langle 2\rangle}$ $\quad x_2 := TAB^{\langle 3\rangle}$ $\quad x_3 := TAB^{\langle 4\rangle}$

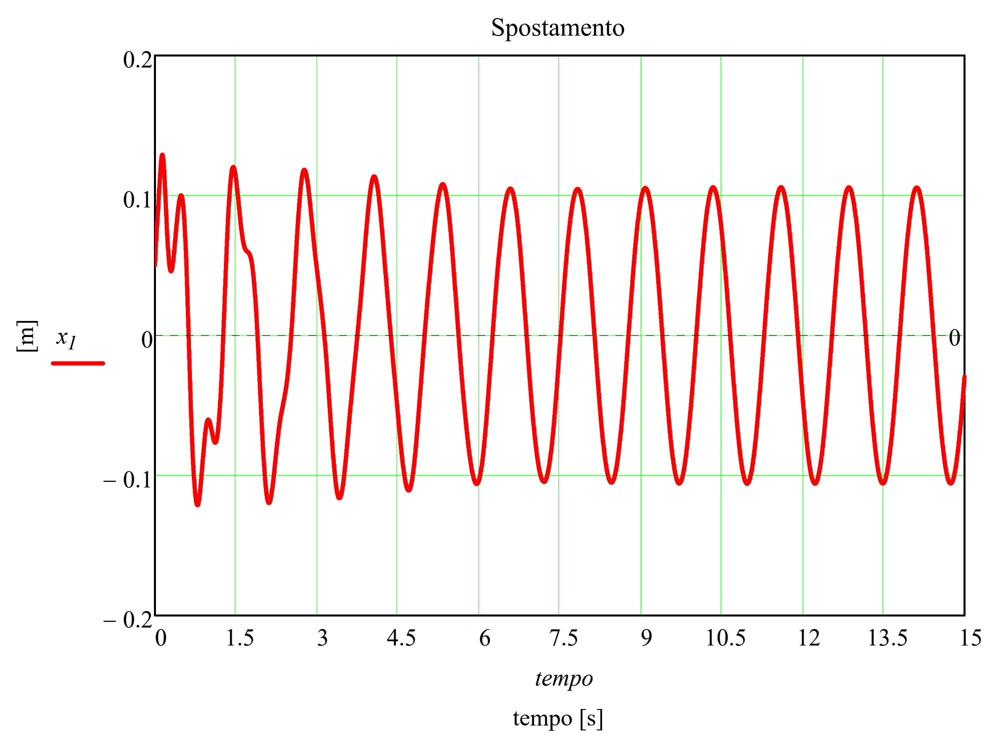
$x'_1 := TAB^{\langle 5\rangle}$ $\quad x'_2 := TAB^{\langle 6\rangle}$ $\quad x'_3 := TAB^{\langle 7\rangle}$

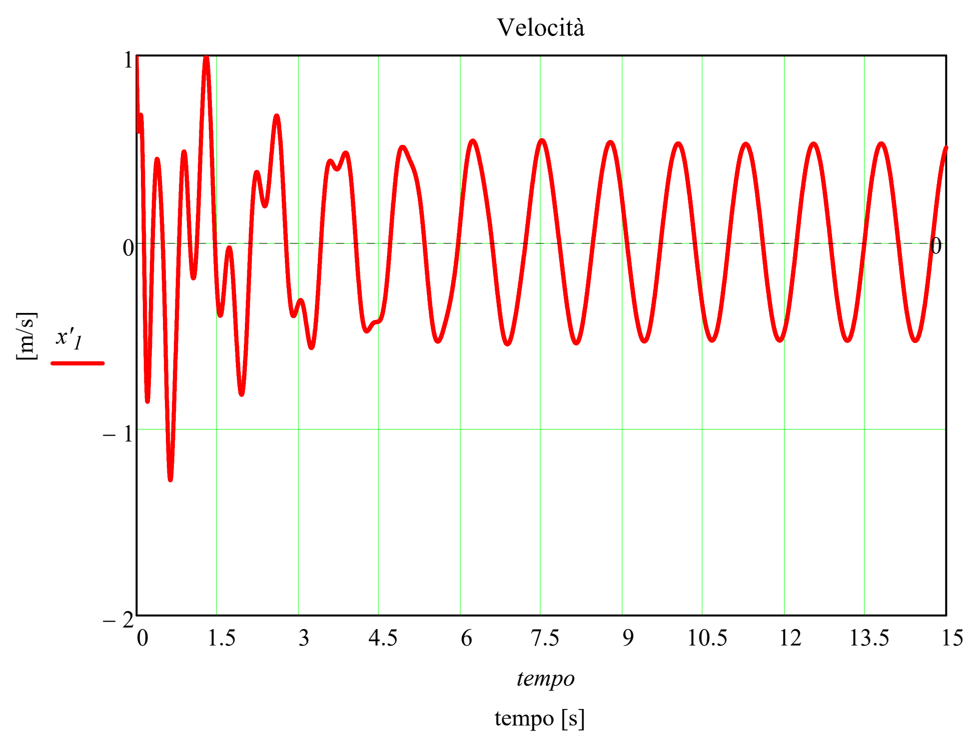
Calcolo delle accelerazioni

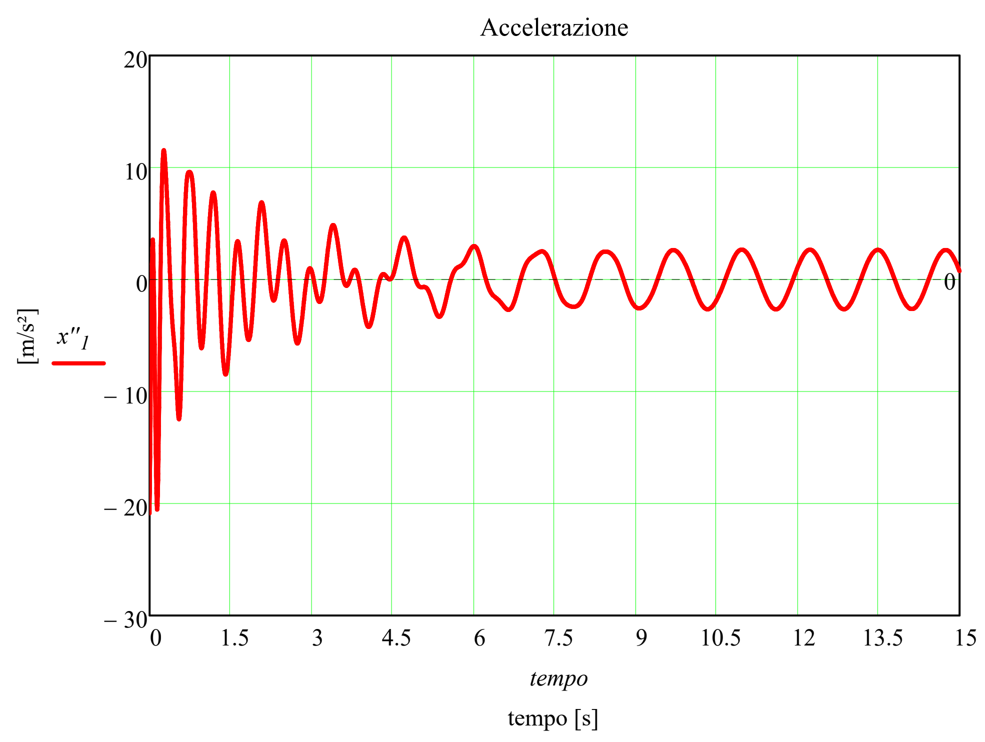
$i := 1..N+1$

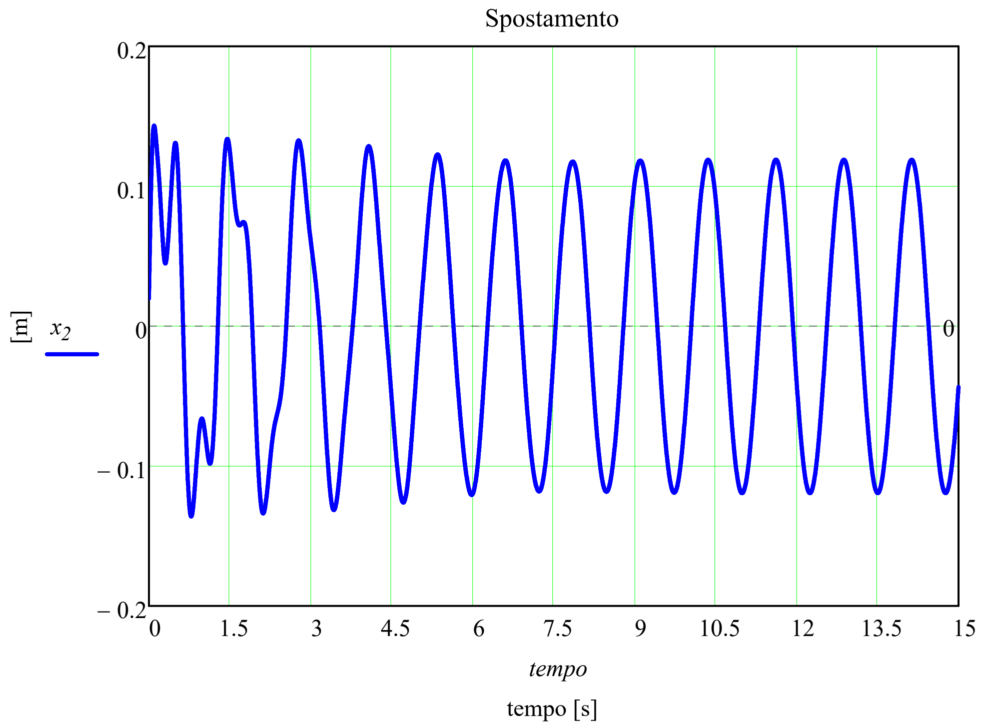
$$x''_{1_i} := \left[\mathbf{A}\cdot\begin{pmatrix} x_{1_i} \\ x_{2_i} \\ x_{3_i} \\ x'_{1_i} \\ x'_{2_i} \\ x'_{3_i} \end{pmatrix} + \mathbf{b}\left(tempo_i\right)\right]_4 \qquad x''_{2_i} := \left[\mathbf{A}\cdot\begin{pmatrix} x_{1_i} \\ x_{2_i} \\ x_{3_i} \\ x'_{1_i} \\ x'_{2_i} \\ x'_{3_i} \end{pmatrix} + \mathbf{b}\left(tempo_i\right)\right]_5 \qquad x''_{3_i} := \left[\mathbf{A}\cdot\begin{pmatrix} x_{1_i} \\ x_{2_i} \\ x_{3_i} \\ x'_{1_i} \\ x'_{2_i} \\ x'_{3_i} \end{pmatrix} + \mathbf{b}\left(tempo_i\right)\right]_6$$

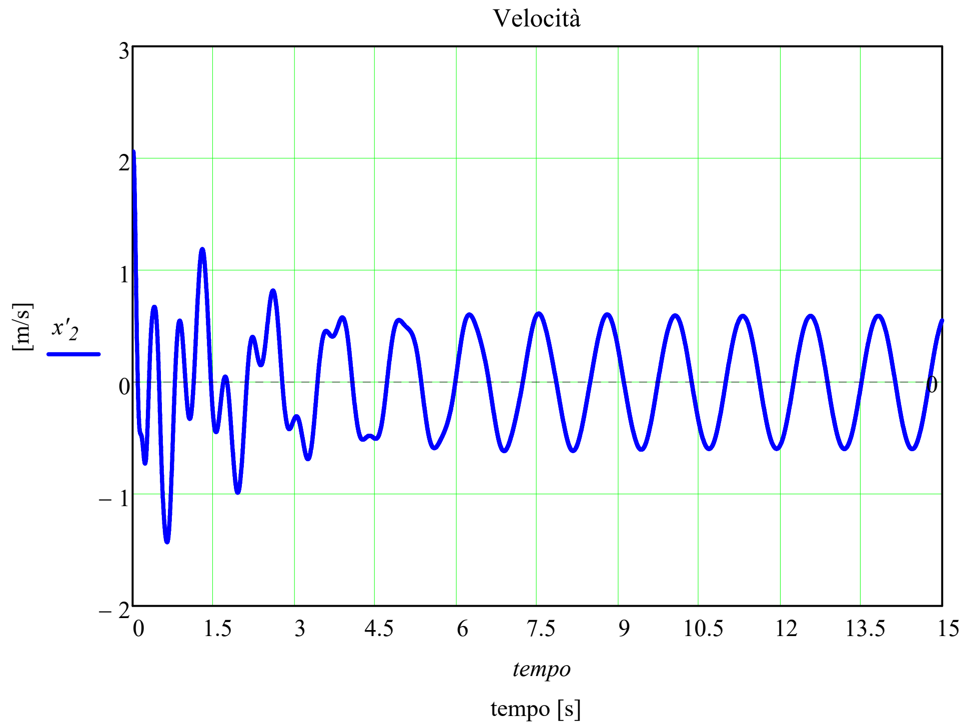
Risultati ottenuti con il metodo di integrazione numerica

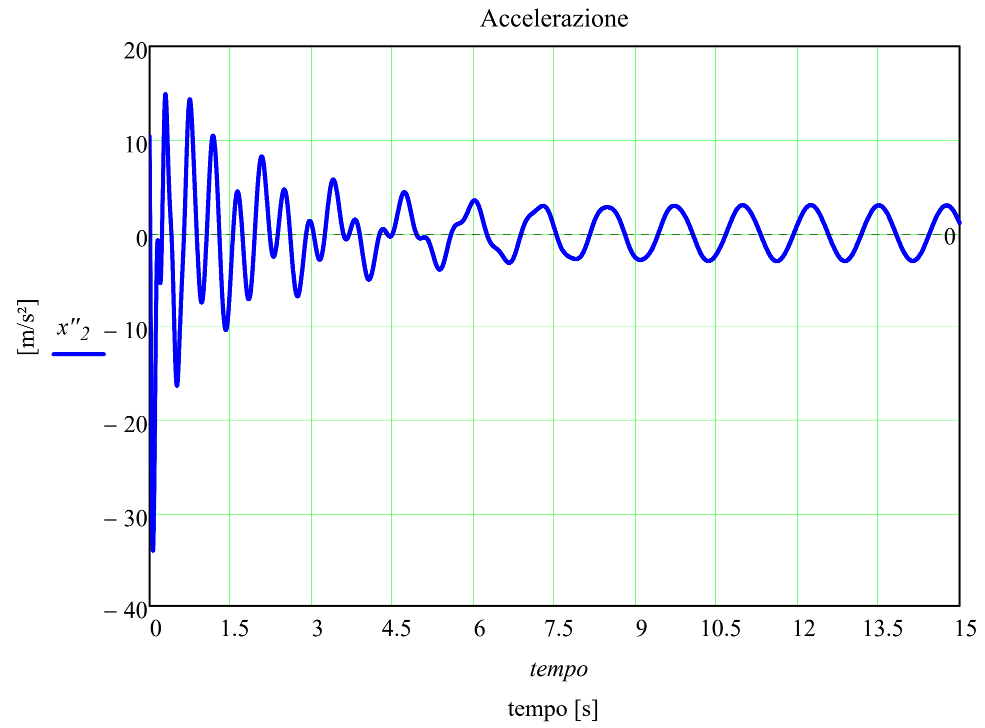


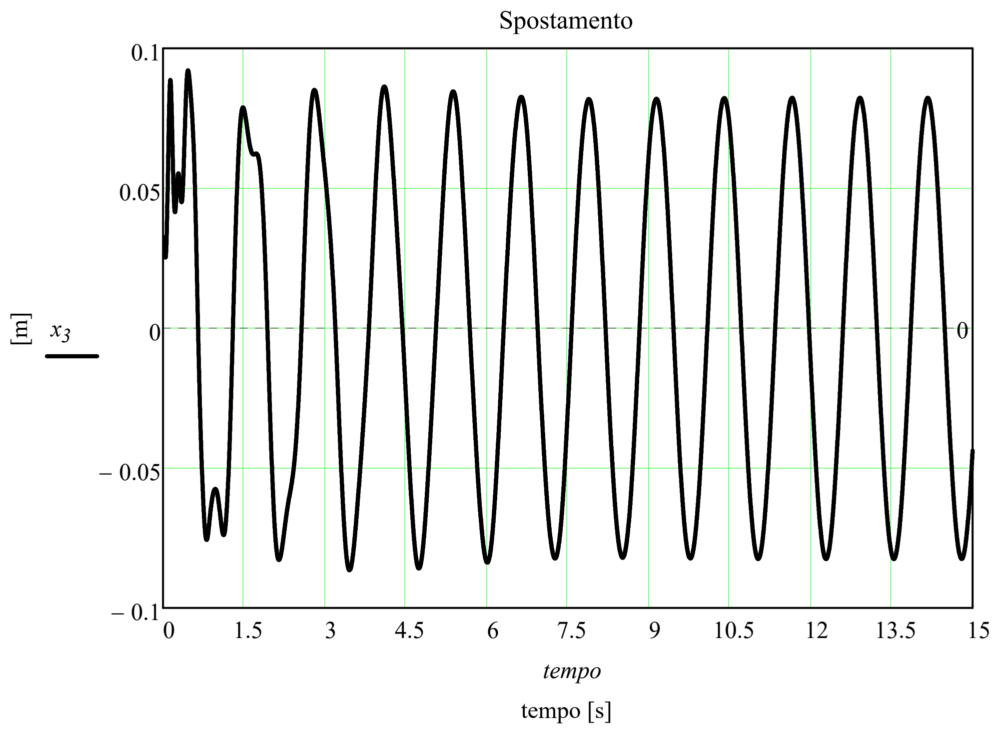


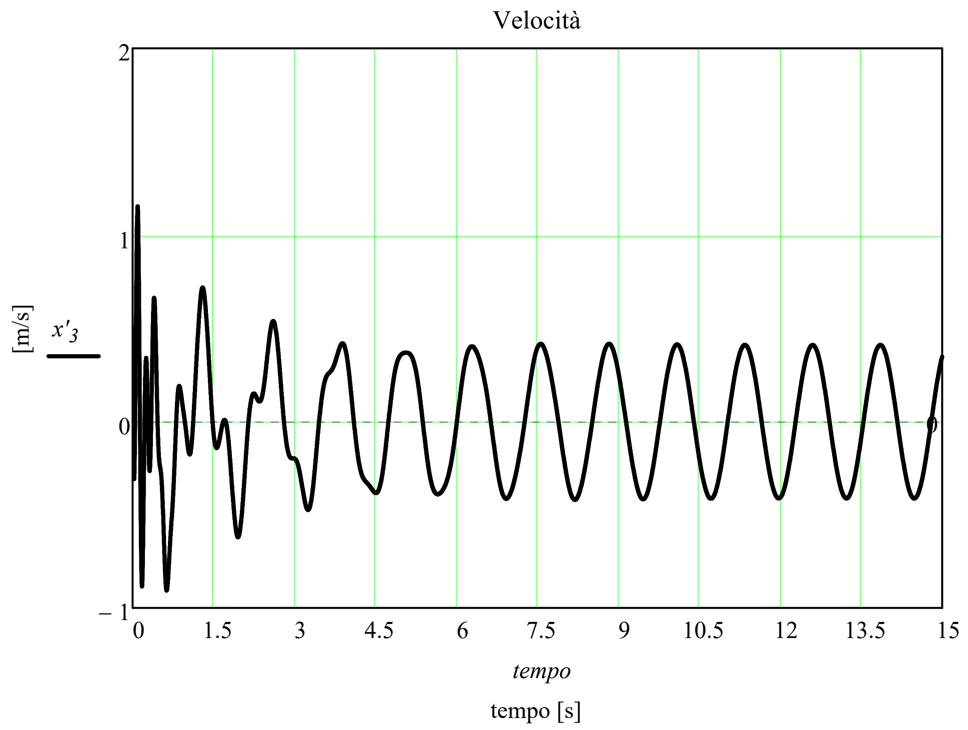


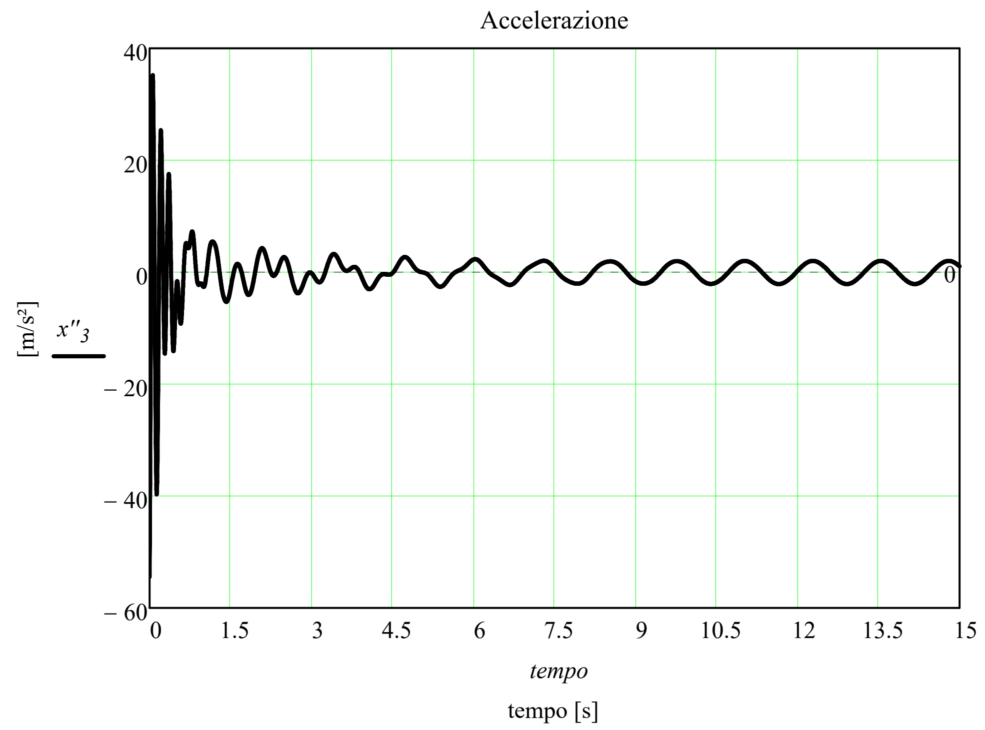












$F_1 = 80$ $\quad F_2 = 60$ $\quad F_3 = 100$

$\psi_1 = 0.349$ $\quad \psi_2 = 0.175$ $\quad \psi_3 = -0.436$ $\quad \Omega = 5$

Matrice di impedenza

$$\mathbf{Z} := \left(\mathbf{K} - \Omega^2 \cdot \mathbf{M}\right) + i \cdot \Omega \cdot \mathbf{C} = \begin{pmatrix} 2425 + 75i & -1500 - 50i & 0 \\ -1500 - 50i & 2775 + 75i & -1400 - 25i \\ 0 & -1400 - 25i & 3150 + 85i \end{pmatrix}$$

$$\mathbf{f} := \begin{pmatrix} F_1 \cdot e^{i \cdot \psi_1} \\ F_2 \cdot e^{i \cdot \psi_2} \\ F_3 \cdot e^{i \cdot \psi_3} \end{pmatrix} = \begin{pmatrix} 75.175 + 27.362i \\ 59.088 + 10.419i \\ 90.631 - 42.262i \end{pmatrix}$$

Risoluzione del sistema lineare, per il calcolo delle ampiezze complesse

$$\mathbf{X} := \mathbf{Z}^{-1} \cdot \mathbf{f} = \begin{pmatrix} 0.105 + 0.013i \\ 0.119 + 3.248i \times 10^{-3} \\ 0.081 - 0.013i \end{pmatrix}$$

Calcolo delle ampiezze e delle fasi per le sinusoidi corrispondenti al moto a regime

$X_1 := \left|\mathbf{X}_1\right| = 0.106$ $\qquad \varphi_1 := arg(\mathbf{X}_1) = 0.119 \cdot rad$

$X_2 := \left|\mathbf{X}_2\right| = 0.119$ $\qquad \varphi_2 := arg(\mathbf{X}_2) = 0.027 \cdot rad$

$X_3 := \left|\mathbf{X}_3\right| = 0.082$ $\qquad \varphi_3 := arg(\mathbf{X}_3) = -0.161 \cdot rad$

$x_{1\_reg}(t) := X_1 \cdot sin(\Omega \cdot t + \varphi_1)$ $\qquad x'_{1\_reg}(t) := \Omega \cdot X_1 \cdot cos(\Omega \cdot t + \varphi_1)$ $\qquad x''_{1\_reg}(t) := -\Omega^2 \cdot X_1 \cdot sin(\Omega \cdot t + \varphi_1)$

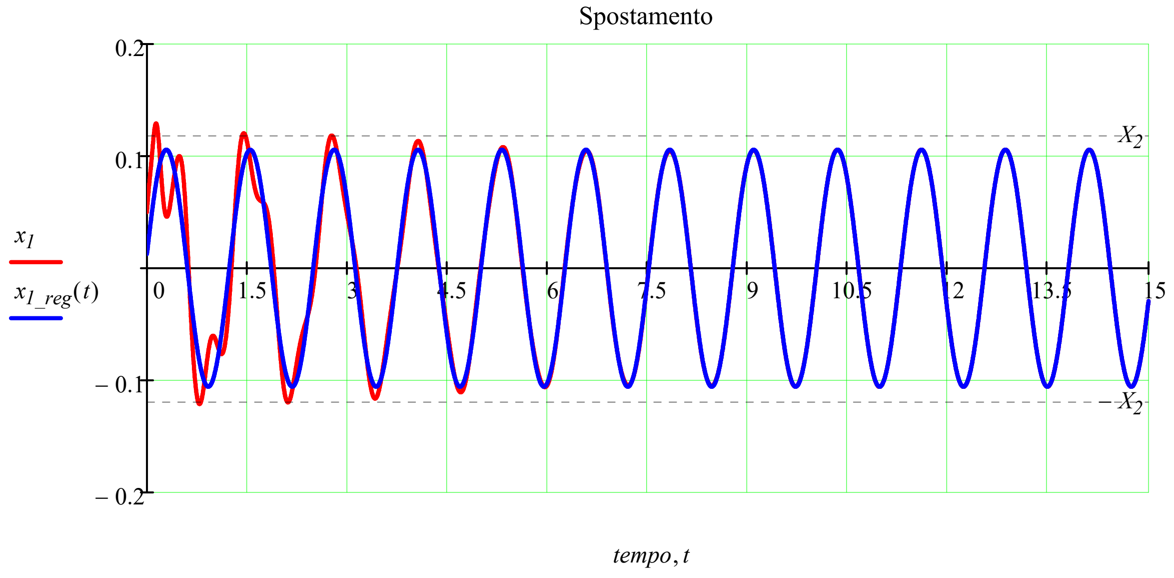
$x_{2\_reg}(t) := X_2 \cdot sin(\Omega \cdot t + \varphi_2)$ $\qquad x'_{2\_reg}(t) := \Omega \cdot X_2 \cdot cos(\Omega \cdot t + \varphi_2)$ $\qquad x''_{2\_reg}(t) := -\Omega^2 \cdot X_2 \cdot sin(\Omega \cdot t + \varphi_2)$

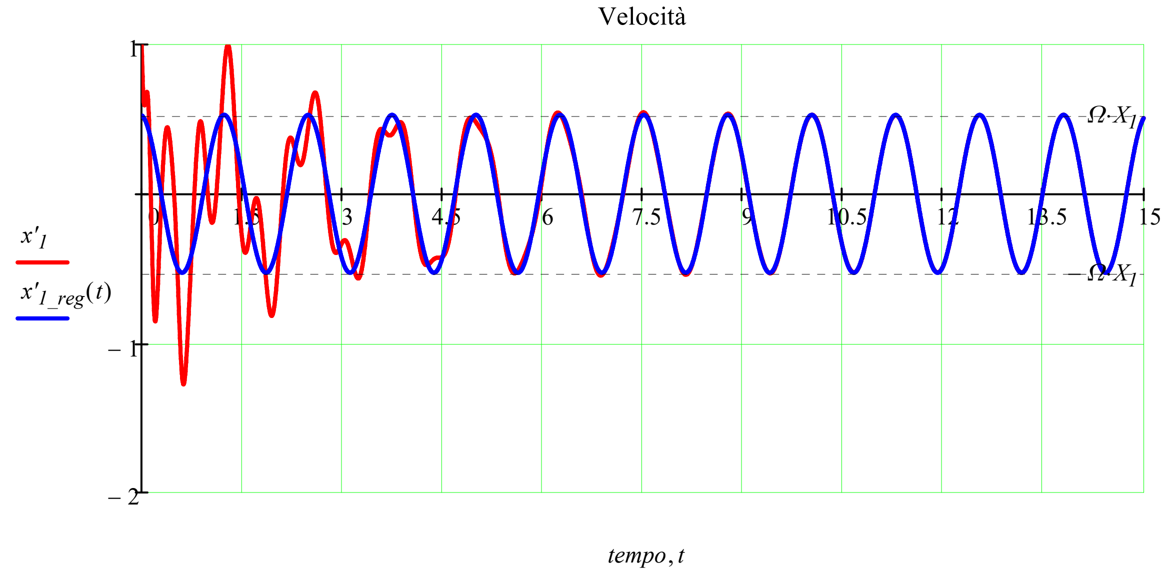
$x_{3\_reg}(t) := X_3 \cdot sin(\Omega \cdot t + \varphi_3)$ $\qquad x'_{3\_reg}(t) := \Omega \cdot X_3 \cdot cos(\Omega \cdot t + \varphi_3)$ $\qquad x''_{3\_reg}(t) := -\Omega^2 \cdot X_3 \cdot sin(\Omega \cdot t + \varphi_3)$

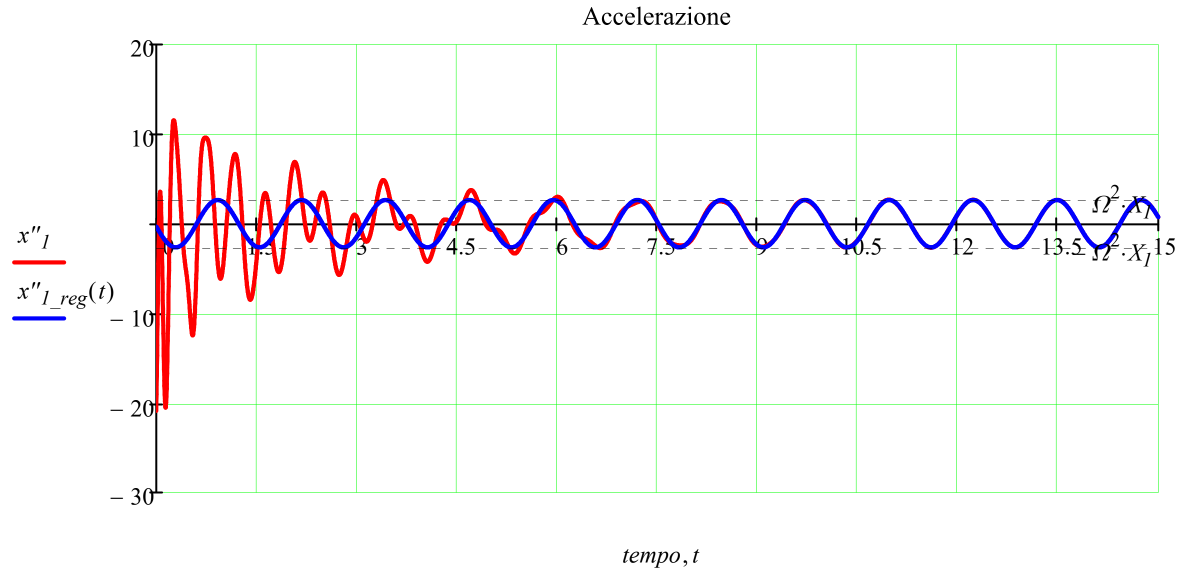
$t := 0, \Delta t \,..\, T_{max}$

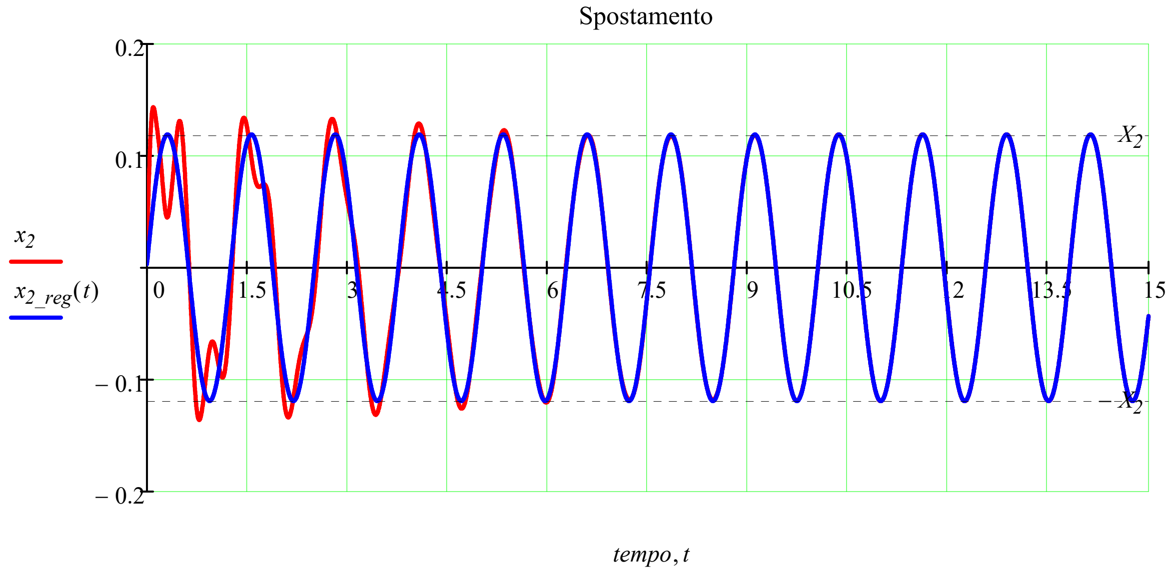
Traccia rossa: soluzione numerica (include sempre il transitorio)
Traccia blu: soluzione di regime (analitica)

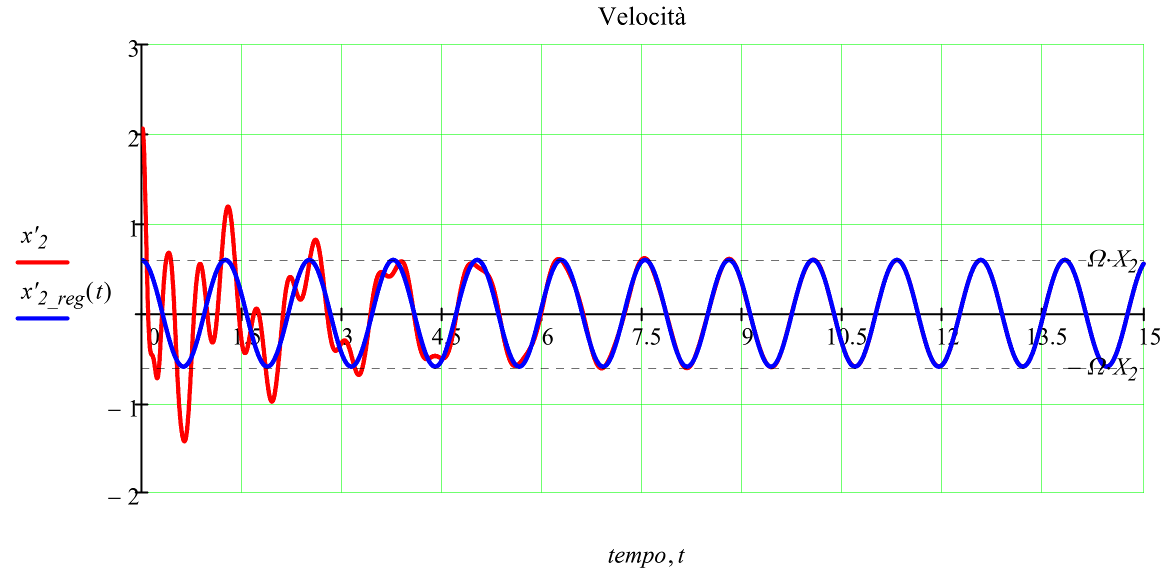
Si nota la differenza all'inizio della vibrazione: infatti la traccia blu, essendo calcolata "a regime", non permette di simulare il transitorio

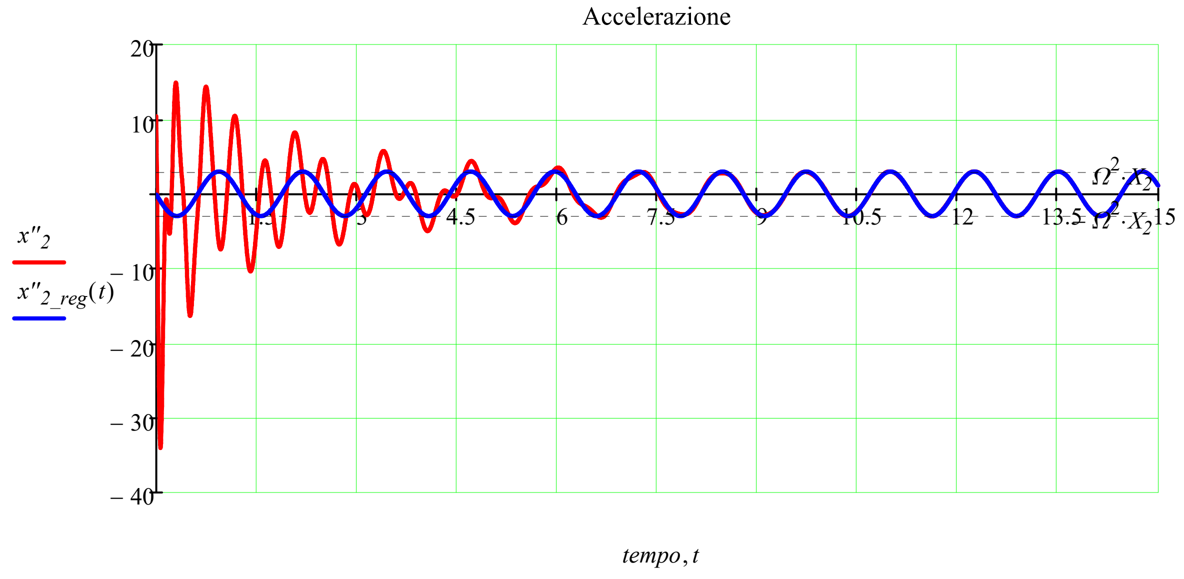


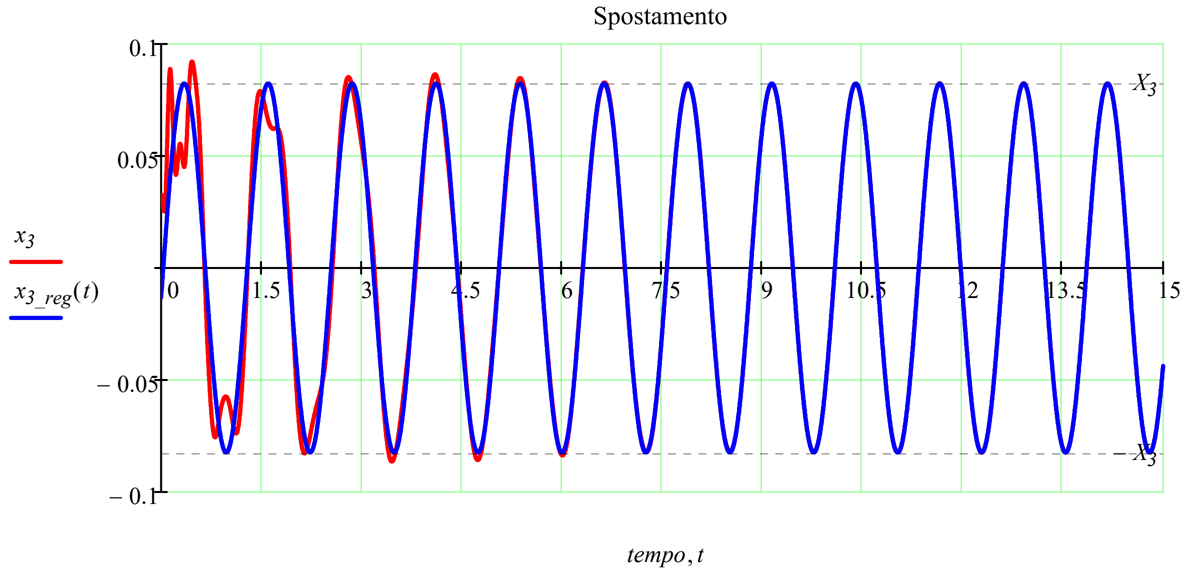


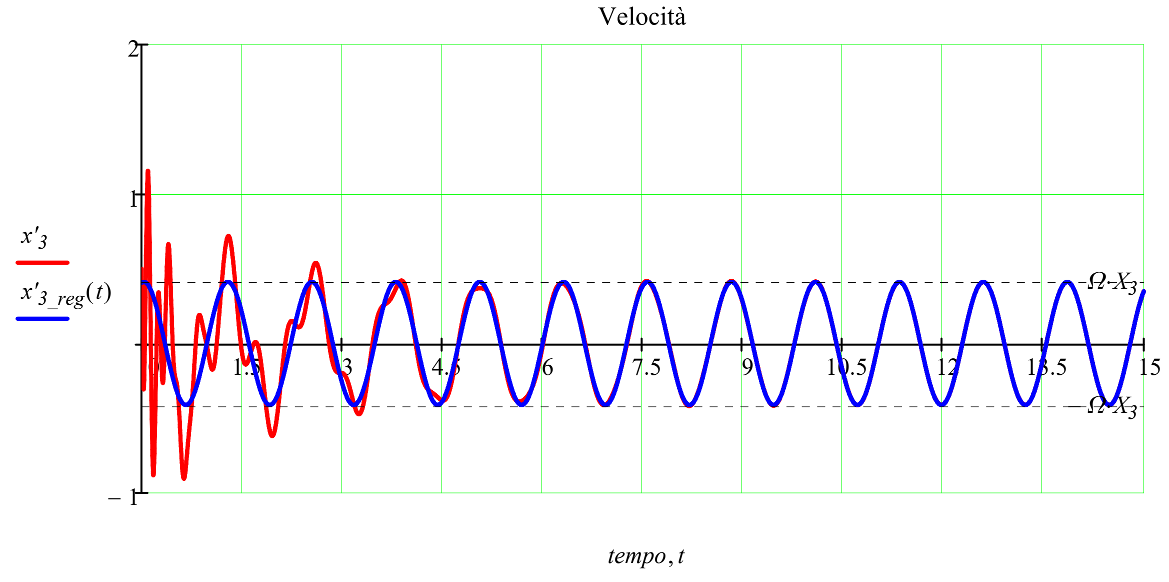


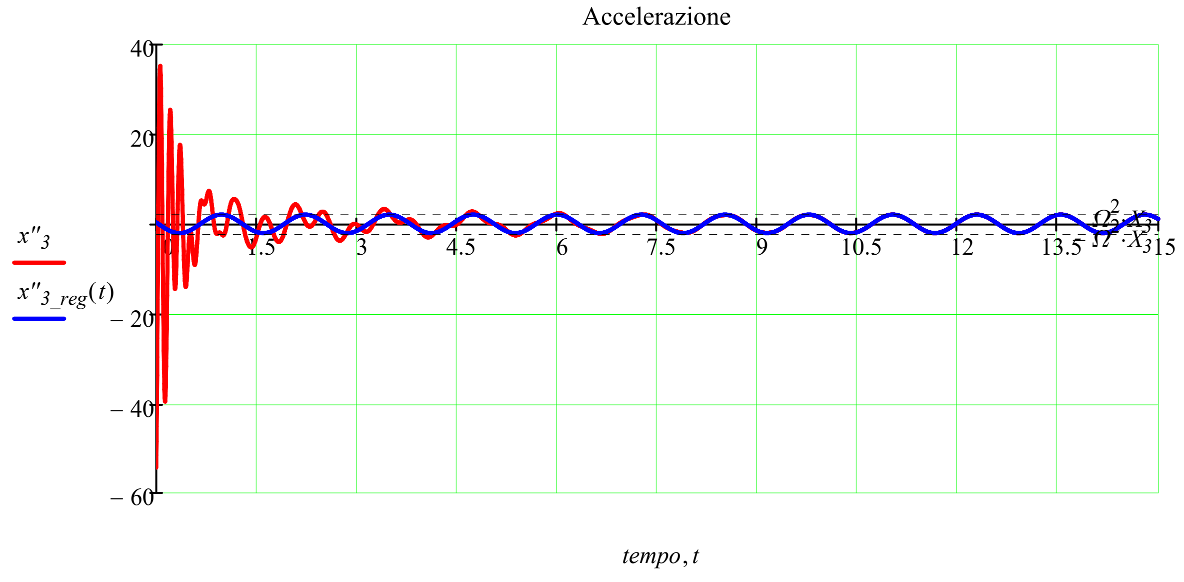












Calcolo del transitorio con metodo analitico e confronto con il metodo numerico

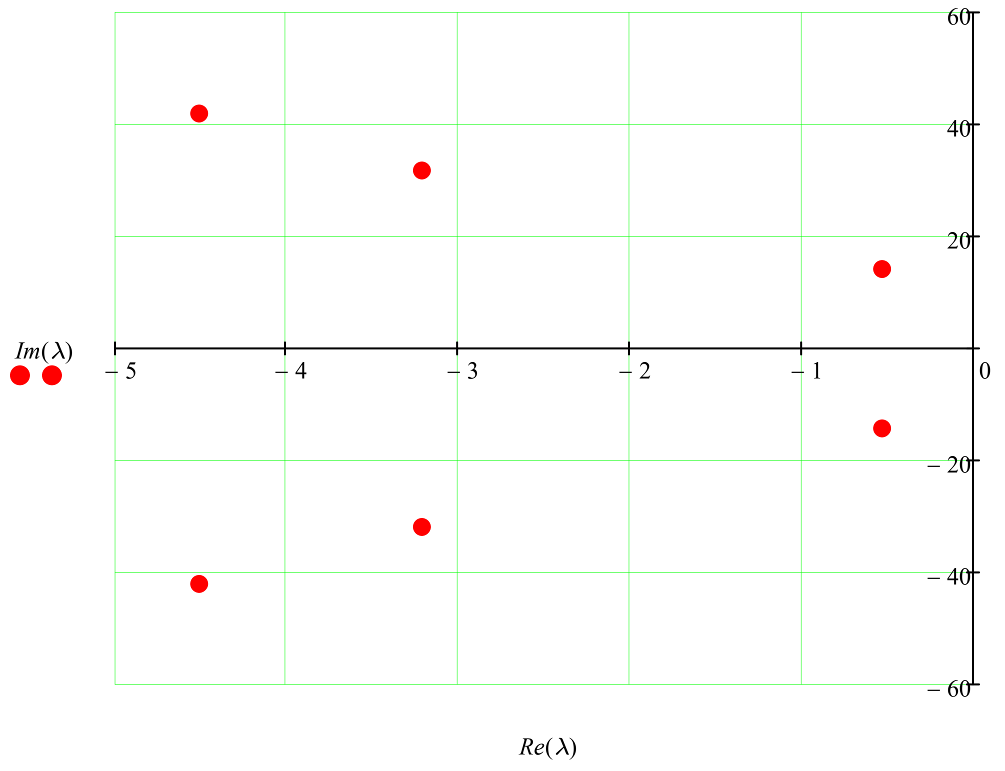
$\mathbf{y}(t) = \mathbf{y}_{omo}(t) + \mathbf{y}_{part}(t)$ Questa formula fornisce la soluzione generale del sistema di equazioni differenziali (vedere Analisi Matematica)

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -833.333 & 500 & 0 & -5 & 3.333 & 0 \\ 300 & -580 & 280 & 2 & -3 & 1 \\ 0 & 700 & -1600 & 0 & 2.5 & -8.5 \end{pmatrix}$$

Matrice di stato

$$\lambda := eigenvals(\mathbf{A}) = \begin{pmatrix} -4.509 + 42.024i \\ -4.509 - 42.024i \\ -3.211 + 31.844i \\ -3.211 - 31.844i \\ -0.53 + 14.21i \\ -0.53 - 14.21i \end{pmatrix} \qquad \sqrt{genvals(\mathbf{K},\mathbf{M})} = \begin{pmatrix} 14.218 \\ 32.007 \\ 42.27 \end{pmatrix}$$

Pulsazioni proprie (sist. non smorzato)



$\mathbf{Y} := eigenvecs(\mathbf{A})$

$\mathbf{Y} =$

	1	2	3	4	5	6	7
1	-0.00012-0.0032i	-0.00012+0.0032i	-0.00268-0.02661i	-0.00268+0.02661i	-0.00127-0.04054i	-0.00127+0.04054i	
2	0.00084+0.00596i	0.00084-0.00596i	0.00078+0.01022i	0.00078-0.01022i	-0.00191-0.05115i	-0.00191+0.05115i	
3	-0.00242-0.02252i	-0.00242+0.02252i	0.00096+0.01242i	0.00096-0.01242i	-0.00159-0.02557i	-0.00159+0.02557i	
4	0.13495+0.00932i	0.13495-0.00932i	0.85582	0.85582	0.57678+0.00346i	0.57678-0.00346i	
5	-0.25444+0.00854i	-0.25444-0.00854i	-0.32786-0.00808i	-0.32786+0.00808i	0.72782	0.72782	
6	0.95725	0.95725	-0.39868-0.00933i	-0.39868+0.00933i	0.36413-0.00907i	0.36413+0.00907i	
7							
8							
9							
10							

Autovettori della matrice di stato, visualizzati come matrice
(la k-esima colonna della matrice è l'autovettore corrispondente al k-esimo autovalore)

Soluzione generale del sistema omogeneo

$$\mathbf{y}_{omo}(t) = \sum_{k=1}^{6} \left(C_k \cdot \mathbf{Y}^{\langle k \rangle} \cdot e^{\lambda_k \cdot t} \right)$$

IMPORTANTE: Le costanti C_k non devono essere calcolate subito !!!

Soluzione particolare del sistema completo (è quella di regime)

$$\mathbf{y}_{part}(t) := \begin{pmatrix} x_{1\_reg}(t) \\ x_{2\_reg}(t) \\ x_{3\_reg}(t) \\ x'_{1\_reg}(t) \\ x'_{2\_reg}(t) \\ x'_{3\_reg}(t) \end{pmatrix} \qquad \mathbf{y'}_{part}(t) := \begin{pmatrix} x'_{1\_reg}(t) \\ x'_{2\_reg}(t) \\ x'_{3\_reg}(t) \\ x''_{1\_reg}(t) \\ x''_{2\_reg}(t) \\ x''_{3\_reg}(t) \end{pmatrix}$$

All' istante iniziale t=0 si avrà:

$$\mathbf{y}(0) = \mathbf{y_{omo}}(0) + \mathbf{y_{part}}(0)$$

Sviluppando i calcoli...

$$\sum_{k=1}^{6}\left(C_k \cdot \mathbf{Y}^{\langle k\rangle}\right) + \begin{pmatrix} X_1 \cdot sin(\varphi_1) \\ X_2 \cdot sin(\varphi_2) \\ X_3 \cdot sin(\varphi_3) \\ \Omega \cdot X_1 \cdot cos(\varphi_1) \\ \Omega \cdot X_2 \cdot cos(\varphi_2) \\ \Omega \cdot X_3 \cdot cos(\varphi_3) \end{pmatrix} = \begin{pmatrix} x_{10} \\ x_{20} \\ x_{30} \\ x'_{10} \\ x'_{20} \\ x'_{30} \end{pmatrix}$$

Le incognite sono le costanti C_k

Possiamo riscrivere le precedenti equazioni nella forma più chiara di sistema lineare: la matrice dei coefficienti di questo sistema è la matrice degli autovettori **Y**

$$\mathbf{Y} \cdot \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \end{pmatrix} = \begin{pmatrix} x_{10} - X_1 \cdot sin(\varphi_1) \\ x_{20} - X_2 \cdot sin(\varphi_2) \\ x_{30} - X_3 \cdot sin(\varphi_3) \\ x'_{10} - \Omega \cdot X_1 \cdot cos(\varphi_1) \\ x'_{20} - \Omega \cdot X_2 \cdot cos(\varphi_2) \\ x'_{30} - \Omega \cdot X_3 \cdot cos(\varphi_3) \end{pmatrix}$$

La risoluzione di questo sistema fornisce le 6 costanti di integrazione

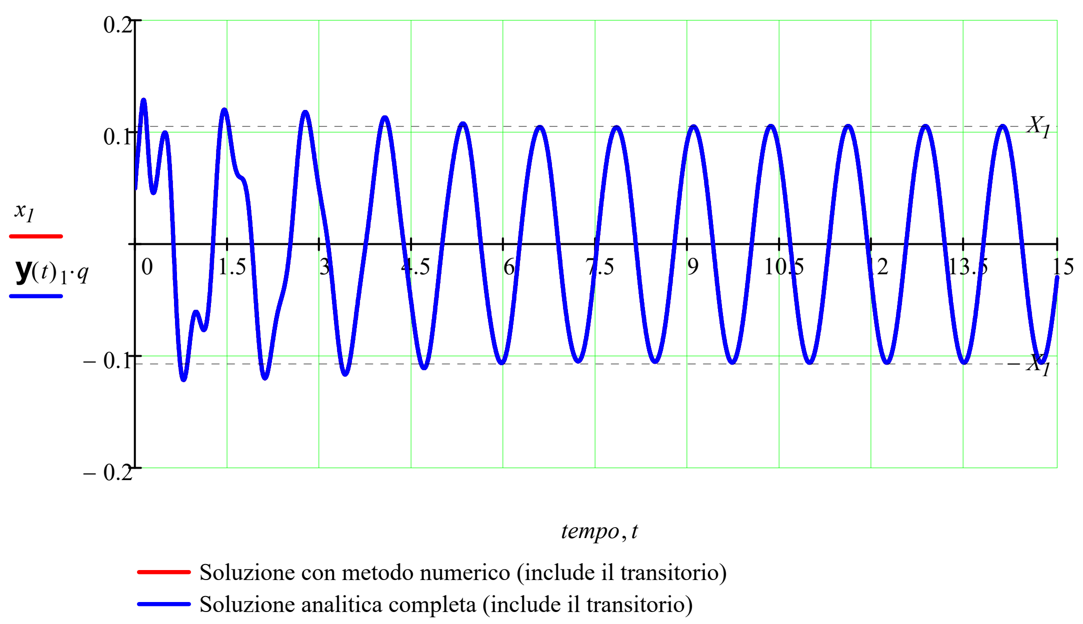
$$C := \mathbf{Y}^{-1} \cdot \begin{pmatrix} x_{10} - X_1 \cdot sin(\varphi_1) \\ x_{20} - X_2 \cdot sin(\varphi_2) \\ x_{30} - X_3 \cdot sin(\varphi_3) \\ x'_{10} - \Omega \cdot X_1 \cdot cos(\varphi_1) \\ x'_{20} - \Omega \cdot X_2 \cdot cos(\varphi_2) \\ x'_{30} - \Omega \cdot X_3 \cdot cos(\varphi_3) \end{pmatrix} \qquad C = \begin{pmatrix} -0.3269 + 0.71498i \\ -0.3269 - 0.71498i \\ -0.18384 + 0.15532i \\ -0.18384 - 0.15532i \\ 0.7748 + 0.31523i \\ 0.7748 - 0.31523i \end{pmatrix}$$

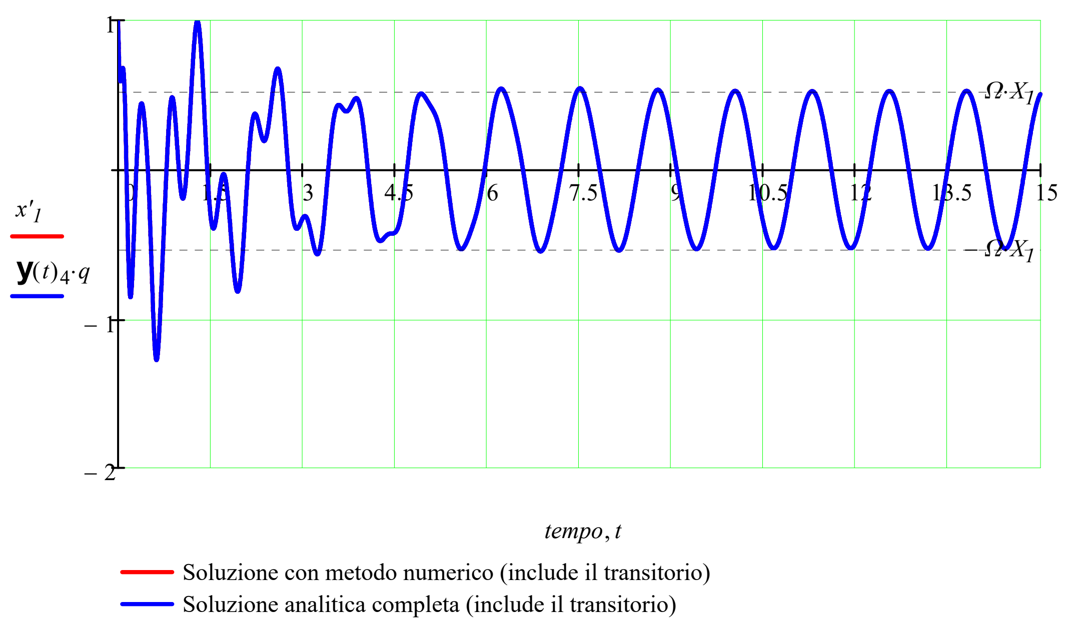
$$\mathbf{y_{omo}}(t) := \sum_{k=1}^{6}\left(C_k \cdot \mathbf{Y}^{\langle k\rangle} \cdot e^{\lambda_k \cdot t}\right) \qquad \mathbf{y'_{omo}}(t) := \sum_{k=1}^{6}\left(\lambda_k \cdot C_k \cdot \mathbf{Y}^{\langle k\rangle} \cdot e^{\lambda_k \cdot t}\right)$$

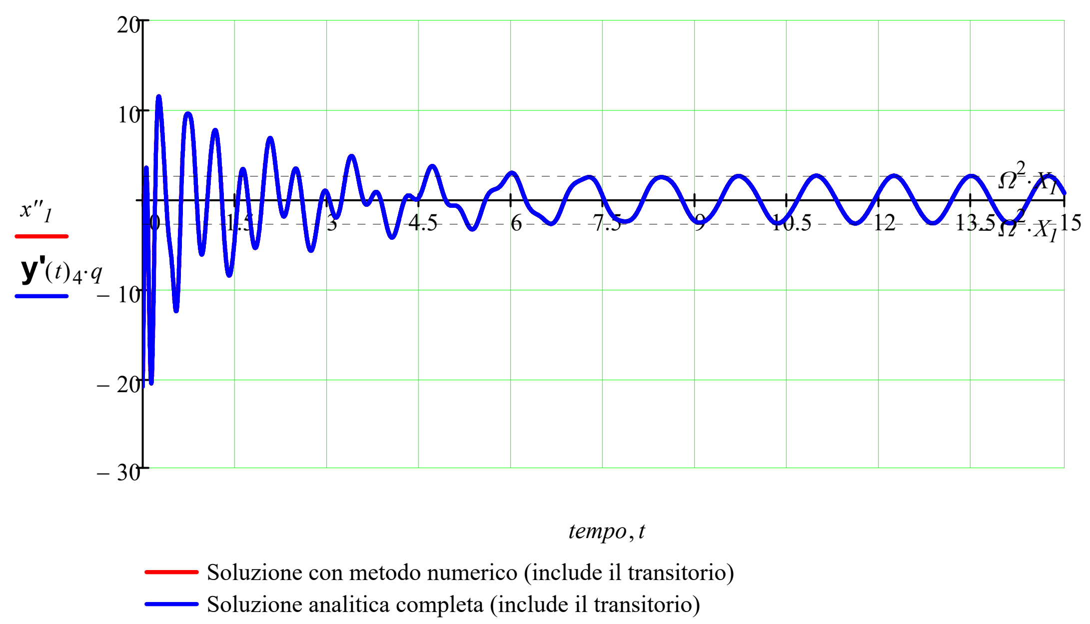
$$\mathbf{y}(t) := \mathbf{y_{omo}}(t) + \mathbf{y_{part}}(t) \qquad \mathbf{y'}(t) := \mathbf{y'_{omo}}(t) + \mathbf{y'_{part}}(t)$$

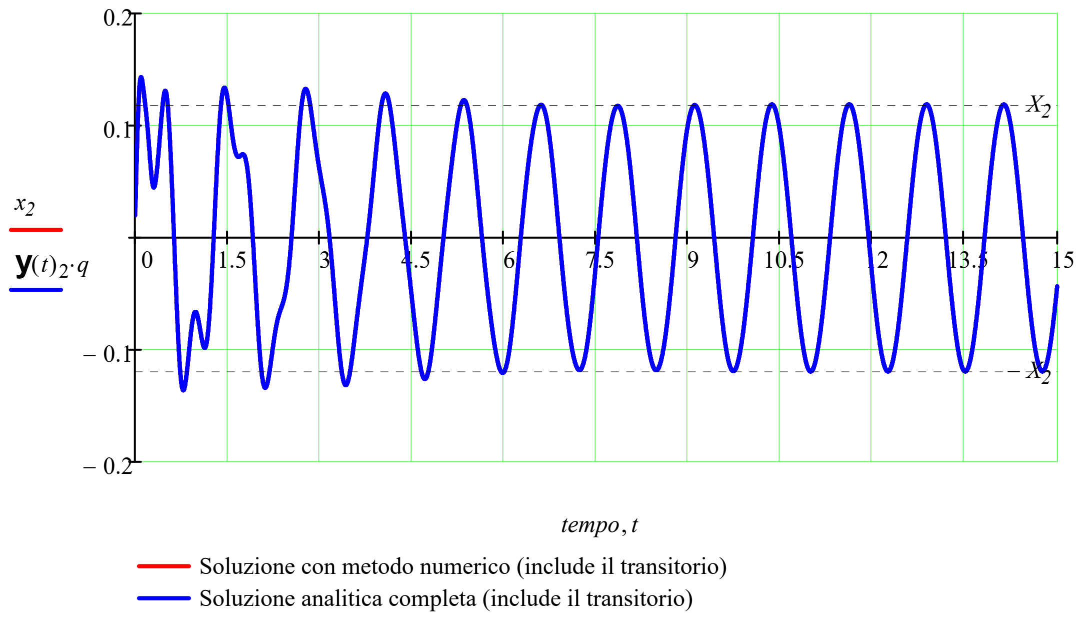
Nei grafici seguenti le tracce rossa e blu si sovrappongono perfettamente

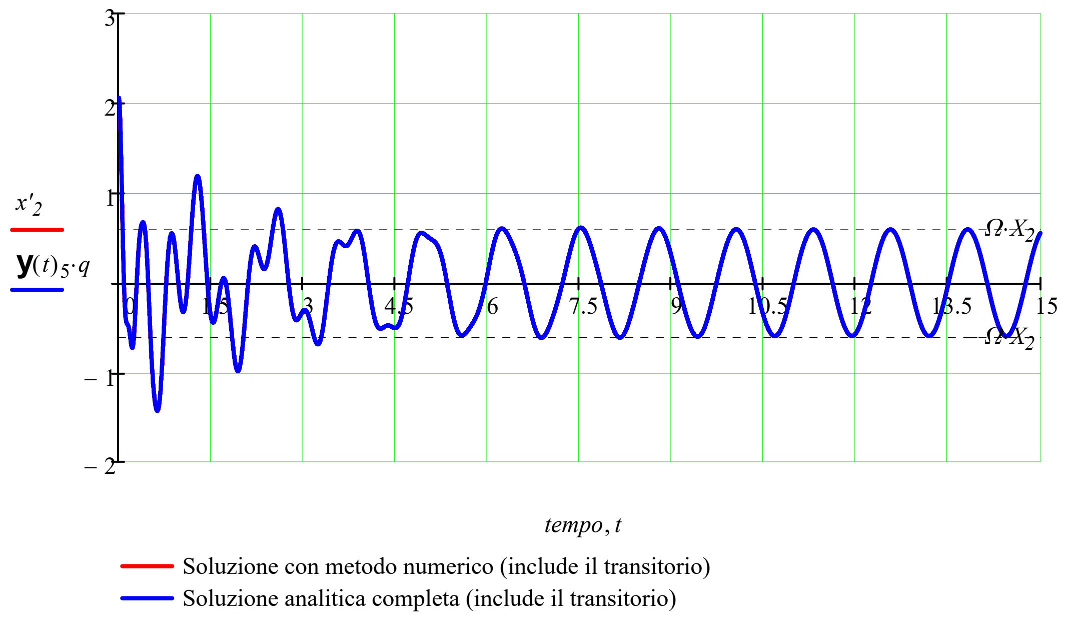
$q := 1$ Comando ON/OFF

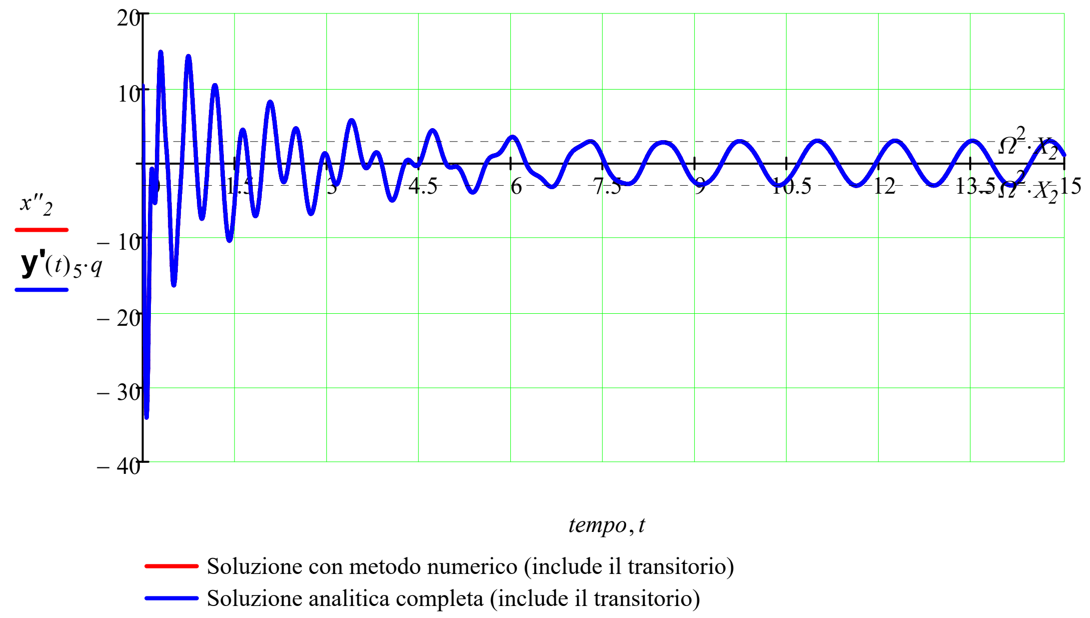


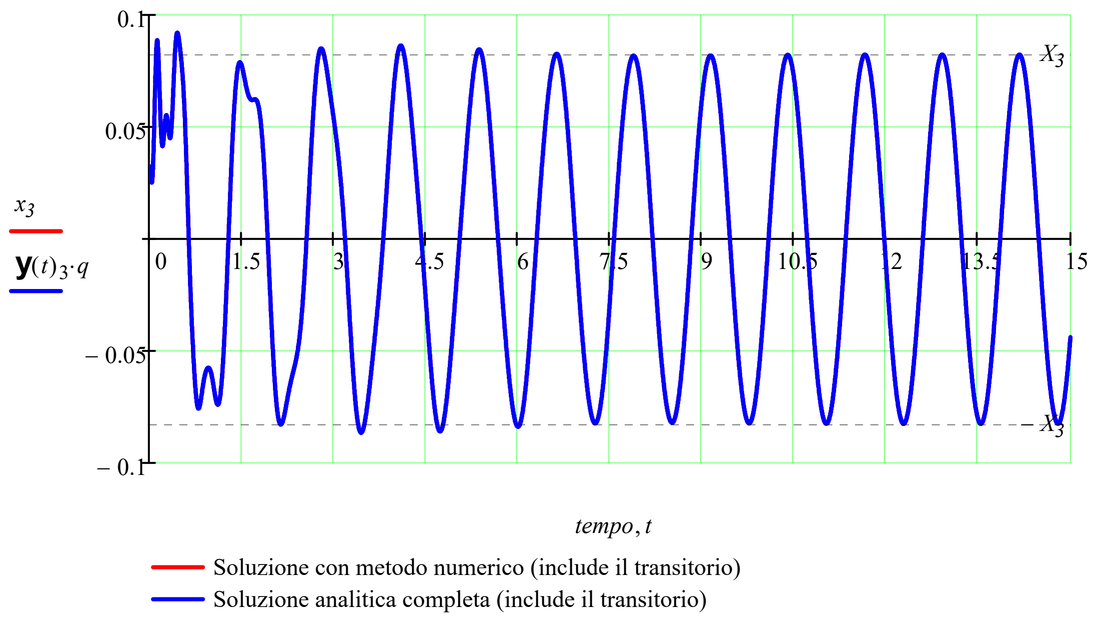


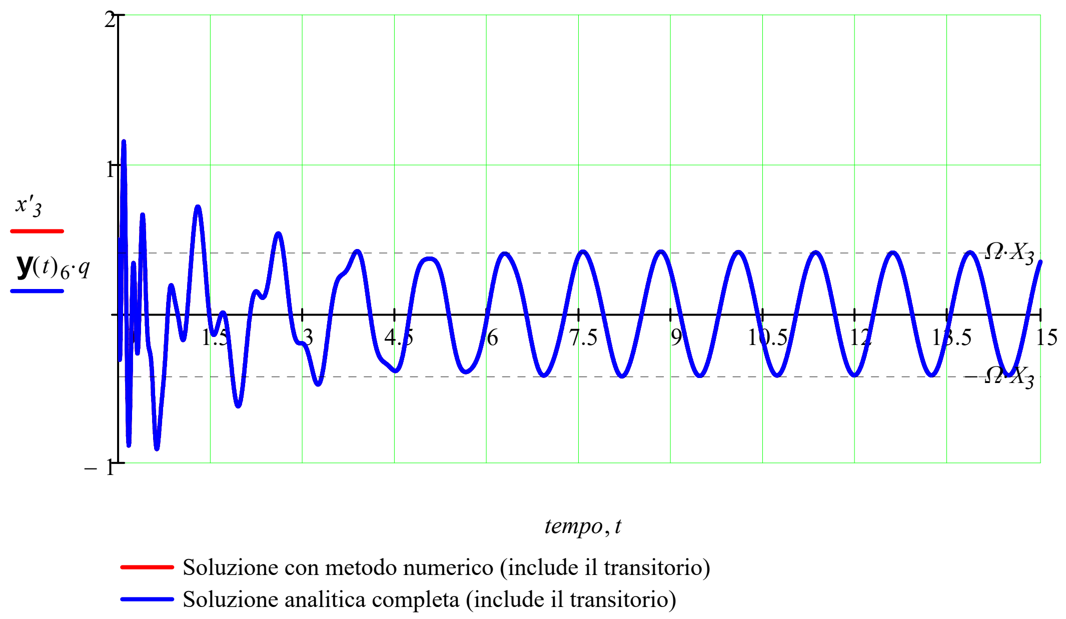


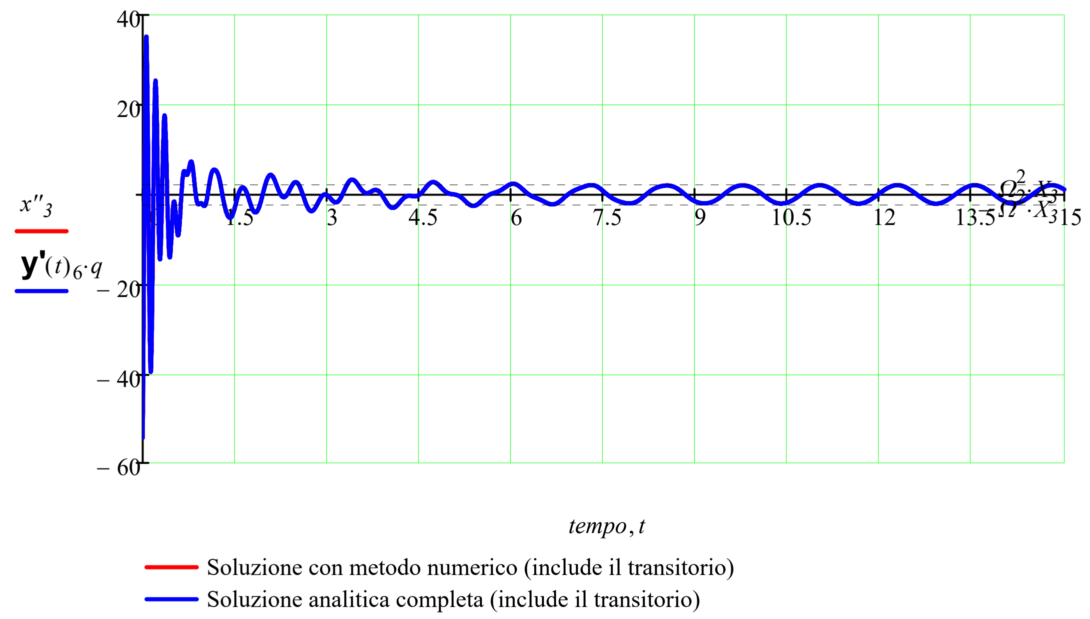

x_2
$\mathbf{y}(t)_2 \cdot q$
X_2
$-X_2$
tempo, t
Soluzione con metodo numerico (include il transitorio)
Soluzione analitica completa (include il transitorio)


x'_2
$\mathbf{y}(t)_5 \cdot q$
$\Omega \cdot X_2$
$-\Omega \cdot X_2$
tempo, t
Soluzione con metodo numerico (include il transitorio)
Soluzione analitica completa (include il transitorio)


x''_2
$\mathbf{y}'(t)_5 \cdot q$
$\Omega^2 \cdot X_2$
$-\Omega^2 \cdot X_2$
tempo, t
Soluzione con metodo numerico (include il transitorio)
Soluzione analitica completa (include il transitorio)


x_3
$\mathbf{y}(t)_3 \cdot q$
X_3
$-X_3$
tempo, t
Soluzione con metodo numerico (include il transitorio)
Soluzione analitica completa (include il transitorio)


x'_3
$\mathbf{y}(t)_6 \cdot q$
$\Omega \cdot X_3$
$-\Omega \cdot X_3$
tempo, t
Soluzione con metodo numerico (include il transitorio)
Soluzione analitica completa (include il transitorio)


x''_3
$\mathbf{y}'(t)_6 \cdot q$
$\Omega^2 \cdot X_3$
$-\Omega^2 \cdot X_3$
tempo, t
Soluzione con metodo numerico (include il transitorio)
Soluzione analitica completa (include il transitorio)